Marko Rucnov

MATH GLOSSARY

(From Elementary Algebra to Mathematical Analysis/Calculus)
Equation Of Us: $MCC = mc^2$

IN THE HONOR OF TWO-HUNDRED YEARS BIRTH OF IRATABA.
MOHAVE CHIEF AND AMERICAN DIPLOMAT, AND CELEBRATING
45 MCC YEARS

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MATH GLOSSARY
With Comments and Solved Problems
(From Elementary Algebra to Mathematical Analysis/Calculus)

1. “Learning without thought is labor lost,
thought without learning is perilous”.
2. “If a man takes no thought about what is distant,
he will find sorrow near at hand”.
3. “I hear and I forget. I see and I remember. I do and I understand”.

Confucius  (551 BC – 479 BC)

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Where There’s Real Knowledge,
There’s Solvent Hope.

Marko Rucnov
“Love your life, perfect your life. Beautify all things in your life.”
Chief Crazy Horse (1842 – 1877)

“Do right always. It will give you satisfaction in life.”
Wovoka (1856 – 1932),
Paiute religious leader, American Indian Prophet

“Trouble no one about their religion; respect others in their view and demand that they respect yours.”
Tecumseh (1768 – 1813),
Native American Shawnee warrior and chief, who became the primary leader of a large, multi-tribal confederacy in the early years of the 19th century

Abel’s* theorem  If a power series \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \) converges at a point \( x_1 \neq x_0 \), then it converges absolutely at any point \( x \) such that \( |x - x_0| < |x_1 - x_0| \). But if series diverges at the point \( x_1 \), then it diverges at any point \( x \) for which \( |x - x_0| > |x_1 - x_0| \).

* Niels Henrik Abel (1802 - 1829), the Norwegian mathematician, at age 19 showed there is no general algebraic solution for roots of a quintic (5th degree) equation or any general polynomial equation of degree greater than four, in terms of explicit algebraic operations (addition, subtraction, multiplication, division, involution, and evolution).
abscissa The first coordinate (x-coordinate) in an ordered pair of numbers. Actually, the abscissa is horizontal coordinate of a point in the coordinate plane (xy-plane).

absolute value (modulus) of a real number \( |n| \) The distance between that number and zero on the number line (representing the nonnegative real number). When we talk about magnitude/modulus of a number or distance between zero and the number, we are talking about absolute value of that number. For example: \( |−5| = 5, |5| = 5, |0| = 0 \).

abundant/excessive number A number whose proper divisors (except for the number itself) have a sum greater than the number. For example 20 is an abundant number because \( 1 + 2 + 4 + 5 + 10 = 22 > 20 \).

ac Method A method for factoring quadratic trinomial \( ax^2 + bx + c \) by grouping using following steps:
1. Factor out the largest common factor, if one exist.
2. Multiply the leading coefficient \( a \) and the constant \( c \).
3. Find a pair of factors of \( ac \) whose sum is \( b \).
4. Rewrite the middle term \( bx \) as a sum or a difference using the factors found in step 3.
5. Factor by grouping.
Example Factor \( 2x^2 + 7x − 15 \).
Solution 1. First, note that there is no common factor.
2. Multiply leading coefficient \( a \) and constant \( c \).
\( ac = 2(−15) = -30 \)
3. Find the pair of factors of \(-30\) so that the sum of these two factors is \( 7 \).

<table>
<thead>
<tr>
<th>Pairs of Factors of -30</th>
<th>Sums of Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, - 30</td>
<td>-29</td>
</tr>
<tr>
<td>-1, 30</td>
<td>29</td>
</tr>
<tr>
<td>2, - 15</td>
<td>-13</td>
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<tr>
<td>-2, 15</td>
<td>13</td>
</tr>
<tr>
<td>-3, 10</td>
<td>7 = b ( \sqrt{ } )</td>
</tr>
</tbody>
</table>

4. We will split middle term \( 7x \) using the result above: \( 7x = -3x + 10x \)
5. Factor by grouping:
\( 2x^2 + 7x − 15 = 2x^2 − 3x + 10x − 15 = x(2x − 3) + 5(2x − 3) = (2x − 3)(x + 5) \).

acceleration The acceleration \( a(t) \) is the derivative of velocity \( v(t) \) with respect to time \( t \). If a body’s position at time \( t \) is \( s = f(t) \), then the body’s acceleration at the time \( t \) is
\[
a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.
\]
accrual  An accrual is an expense or a revenue that gradually increases with the passage of time.

Accumulated Amount of Money Flow at Time  If \( f(x) \) is the rate of money flow at an interest rate \( r \), compounded continuously, at time \( x \), the accumulated amount of money flow at time \( t \) is
\[
A(t) = e^{rt} \int_0^t f(x)e^{-rx} \, dx \quad \text{or} \quad A(t) = e^{rt} \cdot P(t),
\]
where \( P \) is present value of money flow.

Example  The function \( f(x) = 1000x - 100x^2 \) represents the rate of flow of money in dollars per year. Assume a 10-year period at 8% compounded continuously and find the following:
(a) the present value of money flow at \( t = 10 \) and \( r = 0.08 \);
(b) the accumulated amount of money flow at \( t = 10 \).

Solution
(a) The present value \( P(x) \) is given by
\[
P(x) = \int_0^x f(x)e^{-rx} \, dx =
= \left. 1000 \int_0^x xe^{-0.08x} \, dx - 100 \int_0^x x^2 e^{-0.08x} \, dx \right|_0^x
\]
(How to find these two antiderivatives see below (1) and (2))
\[
= \{1000[- (12.5x + 156.25)e^{0.08x}] - 100[- (12.5x^2 + 312.5x + 3906.23)e^{0.08x}]\} |^{10}_0 =
\]
\[
= (-12,500x - 156,250 + 1,250x^2 + 31,250x + 625)e^{-0.08x} =
= (1250x + 18,750x + 234,375)e^{-0.08x} \left|_0^x \right.
\]
\[
= (125000 + 187500 + 156,250)e^{-0.8} - (0 - 0 + 234,375)e^0 \approx 11,351.7775
\]
Thus, the present value is \( P \approx 11,351.78 \)

1) To find the antiderivative \( \int xe^{-0.08x} \, dx \) we will use integration by parts (Method 1: Standard Method).
   Let \( u = x \) and \( dv = e^{-0.08x} \, dx \).
   Then \( du = dx \) and \( v = \int e^{-0.08x} \, dx = \frac{1}{-0.08}e^{-0.08x} = -12.5 \, e^{-0.08x} \).
   So, \( \int x e^{-0.08x}\, dx = x(-12.5e^{-0.08x}) - \int (-12.5e^{-0.08x}) \, dx = -12.5xe^{-0.08x} + \frac{125}{0.08}e^{-0.08x} + C =
\]
\[= - (12.5x + 156.25)e^{-0.08x} + C \quad \ldots \quad (1) \]
2) We will evaluate the antiderivative \( \int x^2 e^{-0.08x} \, dx \) using again integration by parts (Column Integration or Tabular integration, Method 2).
   Choose \( f(x) = x^2 \) as the part to be differentiated (part D), and put \( g(x) = e^{-0.08x} \) in integration column (part I).
   After taking derivatives down the first column and integrals/antiderivatives down the second column we will get the following table below. Next step, draw a diagonal line segment from each term (except the last) in the first (left) column to the term in the next row below in the second (right) column. Label the first diagonal with \( + \), the next with \( - \), and continue alternating the signs. The multiply the terms on opposite ends of each diagonal line segment. Finally, you will sum up the products just formed, adding the \( + \) terms and subtracting the \( - \) terms (See (2)).

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D [f(x) and its derivatives] I [g(x) and its integrals]

\[ x^2 + e^{-0.08x} \]

\[ 2x - 12.5e^{-0.08x}, \quad \text{N.B.} \quad \frac{1}{-0.08} = -12.5; \]

\[ 2 + 156.25e^{-0.08x}, \quad \frac{-12.5}{-0.08} = 156.25; \]

\[ 0 - 1953.125e^{-0.08x}, \quad \frac{156.25}{-0.08} = -1953.125 \]

Thus, \[ \int x^2 e^{-0.08x} \, dx = x^2 (-12.5e^{-0.08x}) - 2x \cdot 156.25e^{-0.08x} + 2(-1953.125e^{-0.08x}) + C = \]

\[ = -(12.5x^2 + 312.5x + 3906.25)e^{-0.08x} + C \quad \text{...} \quad (2) \]

(b) \[ A = e^{0.08 \cdot 10} \int_0^{10} (1000x - 100x^2) e^{-0.08x} \, dx = e^{0.8} \cdot P \approx 25,263.8488. \]

The accumulated value is \$25,263.84.  

Actuarial Method for Unearned Interest  To find unearned interest we will use the formula

\[ u = \frac{n \cdot P \cdot V}{100 + V} \]

where \( u \) is the unearned interest, \( n \) is the number of remaining monthly payments, \( P \) is monthly payment, and \( V \) is the value from the APR table that corresponds to the annual percentage rate for the number of remaining payments (excluding the current payment).

acute angle  An angle measuring between 0° and 90°.

acute triangle  A triangle with all angles less than 90°.

addends/summands  In addition, the numbers/expressions being added.

Adding Real Numbers with the Same Sign  1st step: Add absolute values; 2nd step: Attach their common sign to the sum.

Adding Real Numbers with Opposite Signs  1st step: Take the difference of the absolute values (subtracting the smaller absolute value from the larger absolute value); 2nd step: Attach the sign of the number that has higher absolute value.

addition  Combining numbers/expressions into one sum.

additive identity  The number 0.
additive inverse  A number’s opposite; two numbers are additive inverses of each other if their sum is zero.

The Additive Property of Zero  If \(a\) is a real number, then \(a + 0 = 0 + a = a\).

adjacent angles Two angles in a plane with common vertex and common side but have no interior points in common.

Ahmes Papyrus  The oldest book on mathematics is so called the Ahmes Papyrus from old Egypt (17th century B. C.). Its wonderful title was “Rules for enquiring into nature, and for knowing all that exists, every mystery, every secret”. In this so rare document the unknown quantity is called \(hau\) (heap), and the first problem reads, “Hau, its seventh and its whole makes nineteen”. After this it took approximately 3400 years for symbols we use in algebra to be developed. This word problem we will translate into \(\frac{1}{7}x + x = 19\).

Algebra*  The science of equations and a number of other problems that developed out of the theory of equations. Shortly speaking, algebra is an essential generalization of arithmetic and can be also defined as the science of patterns. The name stemming from the Arabic word “\(Al\ jabr\)” which means -The reunion (of the broken parts) or the equating. The founder/“father” of algebra as a special branch of learning and mathematics was Central Asian/Persian scholar Mohammed ibn-Musa al – Khwarizmi (9th century A. D.) with his famous book ‘\(Al – Kitab al – mukhtasar fi hisab al – jabr wa - l – mugabala\)’/‘The Compendious Book on Calculation by Completion and Balancing’ (written approximately 830 AD).

* Omar Khayyam, Persian mathematician, astronomer, and writer (1048 – 1131): “By the help of God and with His precious assistance, I say that Algebra is scientific art.”

algebraic equation  An equation is called algebraic if each of its members (sides) is a polynomial with respect to the unknown quantities.

algebraic expression  A collection of numbers, variables, operations, and grouping symbols. For example: \(3x^2 + 5xy + 7, (2a^2 + 7b^3)^2, \frac{3x^2 - 5b}{3a+2y}, 5\sqrt{2x}\). In other words, an algebraic expression is an expression in which numbers/constants and letters/variables, are added, subtracted, multiplied, divided, and raised to a natural power, or have their roots extracted.

algebraic fraction/rational expression  A fraction whose numerator and denominator are polynomials. For example, \(\frac{3}{x}, \frac{8ax}{7by}, \frac{5x+y}{3x-2y}, \frac{x^2-4xy+3y^2}{(2x-y)^3}\) are algebraic fractions (or rational expressions).

algebraic function  A function \(f\) is called algebraic if it can be expressed in terms of sums, differences, products, quotients, or roots of polynomial functions.
Algebraic Geometry  Algebraic geometry, as a special branch of mathematics, is the study of solutions to algebraic equations. Generally, algebraic geometry is the study of geometries that come from algebra.

algebraic number  A number is called algebraic if it is a solution of an algebraic/polynomial equation whose coefficients are integers or rational numbers.

algebraic reasoning  Algebraic reasoning is the ability to think logically (See logical thinking) about unknown quantities and the relationships between them.

algebraic structures  Algebraic structures are systems with objects and operations and the rules or properties governing those operations.

algorithm  An algorithm is a series of steps for solving a mathematical problem.

alternate exterior angles  Two nonadjacent exterior angles that lie on opposite sides of a transversal.

alternate interior angles  Two nonadjacent interior angles that lie on opposite sides of a transversal.

altitude  A line segment that is perpendicular to the base of a figure and extends from the base to the opposite vertex, side, or surface.

amicable numbers  Amicable numbers are two different numbers so related that the sum of the proper divisors of each equal to other number. The smallest pair of amicable numbers is \(\{220, 284\}\).

amount of work completed  The amount of work completed \(A\) is given by formula \(A = rt\), where \(r\) is rate of work and \(t\) is amount of time worked.

amplitude  The amplitude of a periodic function is half the difference between the maximum and minimum values of the function.

analytic geometry  A branch of mathematics dealing with geometric properties using algebraic operations and notation to locate points within a coordinate system in one, two or more dimensions. French mathematician and philosopher Rene Descartes (1596-1650) is considered to be the founder of analytic geometry (and modern philosophy: “1. Cogito ergo sum. 2. De omnibus dubitandum est”*).

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*“1. I think, therefore I am. 2. Everything is to be doubted.”

angle  Two rays in a plane with a common endpoint. An angle is formed by rotating a ray around its endpoint.

angle in standard position  An angle is in standard position if its vertex is at the origin of the coordinate plane and its initial side is along the positive x-axis.

angle of depression  The angle of depression from point \(X\) to point \(Y\) (below \(X\)) is the acute angle formed by ray \(XY\) and a horizontal ray with endpoint at \(X\). If you are looking down, it is an angle of depression.
angle of elevation  The angle of elevation from point X to point Y (above X) is the acute angle formed by ray $XY$ and a horizontal ray with endpoint at X. If you are looking up, the angle is an angle of elevation.

angle sum of a triangle  The sum of the measures of the angles of any triangle is $180^0$ (or $\pi$ radians).

Angle-Side-Angle (ASA)  The Angle-Side-Angle (ASA) congruence axiom states that if two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.

angles with orthogonal sides  If two angles have orthogonal/perpendicular sides/legs, then they have the same measure.

angular speed $\omega$  Angular speed measures the speed of rotation and is defined by $\omega = \frac{\theta}{t}$, where $\theta$ is the angle of rotation in radians and $t$ is time. In other words, it is the measure of how fast angle $\theta$ is changing at time $t$.

annuity  A sequence of equal payments made at equal periods of time. In other words, an annuity is an account into which, or out of which, a sequence of scheduled payment are made. If the payments are made at the end of the time period, and the frequency of payments is the same as the frequency of compounding, the annuity is called an ordinary annuity* or a fixed annuity.

* Ordinary Annuity Formula: $A = \frac{p}{\frac{r}{n}} \left[ \left(1 + \frac{r}{n}\right)^{nt} - 1 \right]$, where $A$ is the accumulated amount, of an ordinary annuity with payments of $p$ dollars made $n$ times per year, for $t$ years, at interest rate $r$ (as decimal number).

Example  To save for graduate school, John Smith invests $2000 semiannually in an ordinary annuity with 7% interest compounded semiannually. Determine the accumulated amount in John’s annuity after 10 years.

Solution  Using the given formula for ordinary annuity with $p = $2000, $r = 0.07$, $n = 2$, and $t = 10$, we will get

$$A = \frac{2000\left[1 + \frac{0.07}{2}\right]^{2 \cdot 10} - 1}{0.07/2} = \frac{2000(1.035^{20} - 1)}{0.07} = \frac{4000(1.035^{20} - 1)}{0.07} = $56,559.36$$

annulus  The region between two circles in a plane that have the same center but different radii.

antecedent  See ratio

antiderivative  The function $F(x)$ restored by its derivative or differential is called the antiderivative (or primitive function). In other words, we have to find the function from which the derivative or differential was taken, that is, to solve problem inverse to differentiation. The collection/set of all antiderivatives $F(x)$ of a given function $f(x)$ is called the indefinite integral of $f(x)$ and is denoted by the symbol
\[ \int f(x) \, dx \] This symbol is read: "Integral of \( f(x) \) with respect to \( x \)." Thus, a function \( F(x) \) is antiderivative or primitive function of \( f(x) \) on the interval \([a, b]\) if \( F'(x) = f(x) \), \( \forall x \in [a, b] \).

**Example** Find an equation of the curve whose tangent line has a slope \( f'(x) = x^{\frac{2}{3}} \), given that the point \( \left(1, \frac{3}{5}\right) \) is on the curve.

**Solution**

\[
\frac{df(x)}{dx} = x^{\frac{2}{3}} \rightarrow df(x) = x^{\frac{2}{3}} \, dx \rightarrow f(x) = \int x^{\frac{2}{3}} \, dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C = \frac{3}{5} \sqrt[5]{x^5} + C
\]

\( (1, \frac{3}{5}) \):

\[
\frac{3}{5} = \frac{3}{5} \cdot 1 + C \rightarrow C = 0.
\]

Thus, \( f(x) = \frac{3}{5} \sqrt[5]{x^5} \).

**apex** The point on a geometric figure farthest from the base line or base plane. For example, the apex of a pyramid is its vertex.

**apocalyptic number** A number of the form \( 2^n \) that contains the digits 666* (i.e. the *beast number*) is called an apocalyptic number.

* 666 is the occult "number of the beast", also called "the sign of the devil." The Number of the Beast is a term in the Book of Revelation, of the New Testament.

**apocalypse number** A number having 666 digits (where 666 the *beast number*) is called an apocalypse number.

**applicate** The third coordinate \( z \) in a Cartesian coordinate system in three dimensions.

**Arabic/Indian numerals** The digits used in our base ten (decimal) number system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

**arc** A unbroken part of a circle. In other words, arc is a part of the circumference.

**arc length on a circle** The length \( s \) of the arc intercepted on a circle of radius \( r \) by a central angle of measure \( \theta \) radians is given by the product of the radius and radian measure of the angle, or

\[
s = r\theta, \ \ \theta \ \text{in radians}.
\]

**area** The number of square units that fill a plane figure/region. In other words, this is the number of non-overlapping unit squares of a given size that will exactly cover the interior of a figure.

**area of a sector of a circle** (See sector of a circle) The area \( A \) of a sector of a circle of radius \( r \) and central angle \( \theta \) is given by

\[
A = \frac{1}{2} r^2 \theta, \ \ \theta \ \text{in radians}.
\]
Areas of Surfaces of Revolution If the function $f(x) \geq 0$ is continuously differentiable on the closed interval $[a, b]$, the area of the surface generated by revolving the graph of $y = f(x)$ about $x$-axis is
\[
S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx.
\]

argument 1) Argument is a statement or series of statements for or against something. 2) Argument is one of the independent variables upon whose value that of a function depends.

argument of a complex number When a complex number in standard form $x + yi$ is written in trigonometric form as $r (\cos \theta + i \sin \theta)$, the angle $\theta$ is called the argument of the complex number.

argument of a logarithm In the expression $\log_a x$, $x$ is the argument or logarithmand.

arm (1) One of the legs or sides of a right triangle, not the hypotenuse. (2) One of the rays forming an angle.

arithmetic* The science of numbers and the oldest and most elementary branch of mathematics. The name stemming from the Greek word arithmos which means number.

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*"Arithmetic is the queen of mathematics".
Carl Friedrich Gauss (1777-1855)

arithmetic mean (arithmetic average, or, simply, average) The arithmetic mean/average of $n$ quantities is the sum of these quantities divided by $n$. For example, for $\$16$, $\$22$ and $\$31$ arithmetic mean will be
\[
A. M. = \frac{16+22+31}{3} = \frac{69}{3} = \$23.
\]

arithmetic operations involving fractions (a) Sum of fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$;  
(b) Difference of fractions: $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$;  
(c) Product of fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$;  
(d) Quotient of the division of fractions: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

arithmetic sequence/progression* A sequence of numbers \{a_1, a_2, a_3, \ldots, a_n\} such that the difference any two consecutive terms is a constant called the common difference ($d$). In other words, arithmetic sequence is a sequence of numbers in which each term, except the first term $a_1$, is the result of adding the same number, called the common difference, to the preceding term.

………………
* Formula for the general (nth) term of an arithmetic sequence: $a_n = a_1 + (n-1)d$ 
Formula for the sum, of the first $n$ terms of an arithmetic sequence: $S_n = \frac{n}{2} (a_1+ a_n)$ or $S_n = \frac{n}{2} [2a_1 + (n-1)d]$.

arithmetic series A series for which the associated sequence is arithmetic.
ascending order A polynomial in one variable is written in ascending order if the exponents *increase* from left to right.

ascending sequence A sequence where the numbers are increasing from the beginning to end.

associative property When performing an operation, for example, addition or multiplication, on three or more real numbers, the result is unchanged by the way the numbers are grouped. In other words, the sum or product of three real numbers is the same no matter which two are added or multiplied first.

\[(a + b) + c = a + (b + c) \quad \Rightarrow \quad \text{associative property of addition} \]

\[(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \Rightarrow \quad \text{associative property of multiplication} \]

Astronomical Unit (AU) The *AU* is defined as the average distance between the Earth and the Sun. It is approximately 150 million kilometers (or 93 million miles).

asymptotes of hyperbola The two intersecting straight lines that branches of hyperbola approach are the asymptotes of hyperbola.

asymptotes to the graph of a function (See ‘graphing a rational function’ ) An asymptote to the graph of a function is a straight line such that the distance between the graph (curve) and the line approaches zero as they tend to infinity \( (+\infty \text{ or } -\infty) \). The word asymptote derived from the Greek “asimptotos” which means “not falling together”.

autonomous differential equation A differential equation for which \( \frac{dy}{dt} \) is a function of \( y \) only is called an *autonomous differential equation* – that is the independent variable \( t \) does not appear explicitly, i.e.

\[ \frac{dy}{dt} = f(t). \]

For example, \( \frac{dy}{dt} = (y + 3)(y - 5) \) is an autonomous differential equation.

Notice that an autonomous differential equation is separable and that a solution can be found by integrating \[ \int \frac{dy}{f(y)} = t + C. \]

average The mean of a set of numbers.

average cost If a cost function has the form \( C(x) = mx + b \), where \( x \) is the number of items produced, \( m \) is the marginal cost per item and \( b \) is the fixed cost, then the *average cost* per item is given by equation

\[ \bar{C}(x) = \frac{C(x)}{x} = \frac{mx + b}{x} \]

average rate of change The slope of a straight line gives the average rate of change in \( y \) per unit of change in \( x \), where the value of \( y \) depends on the value of \( x \). See “slope”.

average rate of change of a function over an interval \([x, x + h]\) See “difference quotient.”
average value of a function on the given interval  The average value of a function \( y = f(x) \), integrable on the interval \([a, b]\) is
\[
\text{avg}[f(x)] \text{ or } \bar{f}(x) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

axes/coordinate axes  Two perpendicular number lines (the first line is horizontal and the second line is vertical) used to identify position of points in a coordinate plane. The point \( O(0,0) \) at their intersection is the origin of coordinates. A scale unit is chosen arbitrarily for each axis.

axiom  A basic mathematical principle/theorem not needing a proof. In other words, axiom is an self-evident truth.

Axioms of Congruency of Triangles
1. Angle-Side-Angle (ASA)  \( The \ Angle-Side-Angle \ congruence \ axiom \ states \ that \ that \ if \ two \ angles \ and \ the \ included \ side \ of \ an \ triangle \ are \ equal, \ respectively, \ to \ two \ angles \ and \ the \ included \ side \ of \ a \ second \ triangle, \ then \ the \ triangles \ are \ congruent. \)
2. Side-Angle-Side (SAS)  \( The \ Side-Angle-Side \ congruence \ axiom \ states \ that \ if \ two \ sides \ and \ the \ included \ angle \ of \ one \ triangle \ are \ equal, \ respectively, \ to \ two \ sides \ and \ the \ included \ angle \ of \ a \ second \ triangle, \ then \ the \ triangles \ are \ congruent. \)
3. Side-Side-Side (SSS)  \( The \ Side-Side-Side \ congruence \ axiom \ states \ that \ if \ three \ sides \ of \ one \ triangle \ are \ equal, \ respectively, \ to \ three \ sides \ of \ a \ second \ triangle, \ then \ the \ triangles \ are \ congruent. \)

axiom of parallels  Only one line can be drawn parallel to a given line through a given point not on this line.

axis of symmetry  A straight line that can be drown through a graph such that the part of the graph on one side of the line is an exact reflection of the part on the opposite side.

B

bar graph  A graph that uses parallel bars to compare amounts of the same sort of measurement or to show the frequency of data.

base  In the exponential notation, the number being raised to an exponent. In geometry, the top or bottom of figure.

base of a logarithm  In the expression \( \log_a x \), \( a \) is the base of logarithm.

basic rule of fractions  If the numerator and denominator of a fraction are multiplied by the same non-zero integer or divided by their common factor, a fraction results which is equal to the given fraction, i.e.
\[
\frac{p}{q} = \frac{p \cdot k}{q \cdot k}, \quad k \neq 0.
\]
Beal’s Conjecture* If $a^x + b^y = c^z$, where $a$, $b$, $c$, $x$, $y$ and $z$ are natural numbers and $x$, $y$ and $z$ are all greater than 2, then, $a$, $b$ and $c$ must have a common factor.

* This remarkable conjecture is discovered in 1993 by Texas number theory enthusiast Andrew Beal after his work on famous Fermat’s Last Theorem. Beal’s Conjecture is actually a generalization of Fermat’s Last Theorem, and it was announced later in Mauldin (1997), and cash prize of $1,000,000 has been offered for its proof or counterexample.

Bertrand – Chebyshev theorem/postulate The Bertrand – Chebyshev theorem/postulate states that for any $n > 1$, there exists a prime number $p$ such that $n < p \leq 2n$. This statement was first conjectured in 1845 by Joseph Louis Francois Bertrand (1822 – 1900), French mathematician, and this conjecture was completely proved by Pafnuty Lvovich Chebyshev (1821 – 1894), Russian mathematician.

bearing/angular direction A determination of position of some object in air using angular direction. Bearing is used to identify angles in navigation. One method for expressing bearing uses a single angle, with bearing measured in negative (clockwise) direction from due north. A second method for expressing bearing starts with a north-south line (y-axis) and uses an acute angle to show the angular direction, either east or west, from this line.

betweenness Given three points, $A$, $B$, and $C$, if $AB + BC = AC$, then $A$, $B$, and $C$ are collinear and $B$ is between $A$ and $C$.

biconditional A statement using “if and only if.”

bijection Bijection or bijective function or one-to-one correspondence is a function giving an exact pairing of the elements of two sets (domain and range). Every element of the first set (domain) is paired with exactly one element of the second set (range), and every element of the second is paired with exactly one element of the first set. There are no unpaired elements.

bimodal Having two modes.

binary* number system A number system based on the two digits: 0 and 1.

* In 1998, 17-year old Colin Percival, calculated five trillionth binary digit of $\pi$.

binary operation An operation that can be performed on two and only two elements of a given set. The result is always a single element. In other words, it is an operation that combines two objects of one type to form another object of the same type. For example, addition, subtraction, multiplication, and division are binary operations.

binomial The polynomial that has only two terms. For example, $3a + 5b$. 
**binomial coefficient*** The binomial coefficient is the expression/value \( \frac{n!}{r!(n-r)!} \) for nonnegative integers \( n \) and \( r \). These values are used in calculating the coefficients of the terms of a binomial expansion \( (a + b)^n \).

* Girolamo Cardano (1501 – 1576) was an Italian polymath and earliest introducer of the binomial coefficients and the binomial theorem.

**binomial experiment** An experiment consisting of \( n \) independent trials, with two possible outcomes for each trial, and constant probabilities for each trial.

**Binomial Probability** If \( p \) is the probability of success on a single trial of a binomial experiment and \( q \) is the probability of failure on a single trial, the probability of \( x \) successes and \( n - x \) failures in \( n \) independent repeated trials of the experiment, known as binomial probability, is

\[
P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}, \text{ where } nC_n \text{ is the number of combinations when } x \text{ objects are selected from } n \text{ objects, and } p + q = 1.
\]

**Example** The probability that a birth will result in twins is 0.012. Assuming independence (perhaps not a valid assumption), what are the probabilities that out of 100 births in a hospital, there will be exactly 2 sets of twins?

**Solution** We have here \( n = 100, p = 0.012, x = 2, 1 - p = 0.988 \). Thus,

\[
P(\text{exactly 2 sets of twins}) = \frac{100!}{2!(100-2)!} (0.012)^2 (0.988)^98 \approx 0.2183
\]

**binomial theorem (general binomial expansion)** The binomial theorem is used to expand a binomial \( (a + b)^n \) in the form of a polynomial for positive integer \( n \). The binomial formula for \( n \)th power of \( a + b \) is:

\[
(a + b)^n = \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + b^n =
\]

\[
= a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \ldots + \frac{n!}{(n-1)!1!}ab^{n-1} + b^n.
\]

N.B. The factorial notation “\( n! \)” means “the product all natural numbers between 1 and \( n \)” or \( n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n \).

**The \( k \)th term** of the binomial expansion \( (a + b)^n \) is given as follows

\[
\binom{n}{k} a^{n-(k-1)} b^{k-1}, \text{ where } n \geq k - 1.
\]

**binormal vector** The binormal vector \( B \) is given by vector (or cross) product of the unit tangent vector \( T \) and unit normal vector \( N \), i.e.

\[
B(t) = T(t) \times N(t)
\]

The binormal vector \( B \) is also unit vector. Together, \( T, N \) and \( B \) are mutually orthogonal unit vectors that can be drawn from the position vector \( r(t) \) as a reference point. These three unit vectors also define
three planes. The plane defined by \( T \) and \( N \) is called the \textit{osculating plane}. The plane defined by \( N \) and \( B \) is called the \textit{normal plane}. The plane defined by \( T \) and \( B \) is called the \textit{rectifying plane}.

**biquadratic* equation** The equation of the form \( ax^4 + bx^2 + c = 0 \). It can be reduced to a quadratic equation by introducing a new unknown thus: \( x^2 = u \).

*Any equation of the form \( ax^{2n} + bx^n + c = 0 \) can be also reduced to a quadratic equation by introducing a new unknown \( u = x^n \).

**bisect** To cut or divide into two equal parts.

**bisector** The straight line that divides an angle into equal pieces.

**Bolzano-Weierstrass theorem** Any bounded sequence always contains a convergent subsequence. This theorem named after Bernard Bolzano (1781–1848) and Karl Weierstrass (1815–1897), is fundamental result about convergence in a \( n \)-dimensional Euclidian space.

**book value** The value of the equipment after certain time. For example, if a $13,000 car depreciates $2,000 per year, the book value after 3 years is $13,000 - 3 \cdot 2,000 = $13,000 - $6,000 = $7,000.

**boundary (line)** A line which separates the plane into two half-planes.

**bounded sequence** The sequence \( \{ a_n \} \) is said to be bounded if \( \exists M > 0 \ \exists n : |a_n| \leq M \).

**bounded set** The set \( X \) is \textit{bounded from above (from below)} if there is a number \( M \) (\( m \)) such that for any number \( x \) from the set \( X \) the inequality \( x \leq M (x \geq m) \) holds true. The number \( M \) (\( m \)) is the \textit{upper (the lower) bound} of the set \( X \).

**Boundedness of Functions** A function \( y = f(x) \), \( x \in A \), is said to be bounded if there is a number \( M \) such that for any \( x \in A \) the following inequality is satisfied
\[
|f(x)| \leq M. \quad (1)
\]
Otherwise, the function is called unbounded. Geometrical condition (1) means that the graph of bounded function \( y = f(x) \) is located on the coordinate plane in the horizontal band \( -M \leq y \leq M \).

**Boundedness Theorem** Let \( f(x) \) be a polynomial function of degree \( n \geq 1 \) with real coefficients and with a positive leading coefficient. If \( f(x) \) is divided synthetically by \( x - c \), and
(a) if \( c > 0 \) and all numbers in bottom row of the synthetic division are nonnegative, then \( f(x) \) has no zero (x-intercept) greater than \( c \); 
(b) if \( c < 0 \) and numbers in the bottom row of the synthetic division alternate in sign (with 0 considered positive or negative, as needed), then \( f(x) \) has no zero (x-intercept) less than \( c \). 
This theorem shows how the bottom row of a synthetic division can be used to place upper and lower bounds on possible real zeros (x-intercepts) of a polynomial function.

**Example** Show that the real zeros of the polynomial function satisfy the given condition.
\[ f(x) = 2x^5 - x^4 + 2x^3 - 2x^2 + 4x - 4; \text{ no real zero greater than 1.} \]

**Solution** Since \( f(x) \) has real coefficients and the leading coefficient, \( 2 \), is positive, we will use the boundedness theorem and we will divide \( f(x) \) synthetically by \( x - 1 \).

\[
\begin{array}{c|cccc}
1 & 2 & 1 & 3 & 1 & 5 \\
\hline
2 & 1 & 3 & 1 & 5 & 1 & \leftarrow \text{All are nonnegative.}
\end{array}
\]

Since \( 1 > 0 \) and all numbers in the last row are nonnegative, \( f(x) \) has no zero greater than 1.

**break-even quantity\/point** The number of units of which **revenue** just equals **cost** (the **profit** is 0): the corresponding ordered pair gives the **break-even point**. In other words, the break-even point is the point of the intersection of the graphs of the revenue function and the cost function.

\[ R(x) \geq C(x) \]

Example 1 If the revenue and cost function of a picture frames are given by

\[ R(x) = 60x \quad \text{and} \quad C(x) = 50x + 5000, \]

where \( x \) is the number of picture frames produced and sold, at what production level does \( R(x) \) at least equal \( C(x) \)?

**Solution** The product will at least break even when \( R(x) \geq C(x) \). We will set \( R(x) \geq C(x) \) and solve this inequality for \( x \).

\[ R(x) \geq C(x) \quad \Rightarrow \quad 60x \geq 50x + 5000 \quad \Rightarrow \quad x \geq 500 \]

The break-even point is at \( x = 500 \). This product will at least break even if the number of units of picture frames produced and sold is in the interval \([500, \infty)\).

Example 2. If the cost to produce \( x \) units of coffee cups is \( C = 105x + 900 \), while the revenue is \( R = 85x \), at what production level does \( R \) at least equal \( C \) (or where product will at least break even)?

**Solution** We will set \( R(x) \geq C(x) \) and solve this inequality for \( x \).

\[ R(x) \geq C(x) \quad \Rightarrow \quad 85x \geq 105x + 900 \quad \Rightarrow \quad -20x \geq 900 \quad \Rightarrow \quad x \leq -45 \quad \Rightarrow \quad \text{The product will never break even because } x < 0. \]

Calculus/Mathematical Analysis

Calculus, as one branch of Mathematics, is the study of ‘Rates of Change’. There are two main branches of Calculus: Differential Calculus and Integral Calculus. Differential Calculus determines the rate of change* of the given function/quantity, Integral Calculus finds the function/quantity where the rate of change is known.

\[ \text{-------------} \]

*1) **Stephen Hawking** (1942 – 2018), Explorer of Black Holes: “Intelligence is the ability to adapt to change.”
2) **Bob Dylan** (1941), American musician and writer: “*There is nothing so stable as change.*”

**cardinal number** Any whole number that expresses the number of objects, items, elements or units under consideration. In other words, the cardinal number of the set $A$ is a whole number $n(A)$ that names how many elements/objects are in set $A$.

**cardioid** A cardioid is a heart-shaped curve that is the graph of a polar equation of the form $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$, where $\left| \frac{a}{b} \right| = 1$.

**Cartesian plane** The $x,y$ coordinate plane is often called the Cartesian plane because French mathematician and philosopher **Rene Descartes Cartesius** first introduced it in 1637.

**Cartesian Product** The *Cartesian product* of set $A$ and set $B$, symbolized by $A \times B$ and read “$A$ cross $B$”, is the set of all possible ordered pairs of the form $(a, b)$, where $a \in A$ and $b \in B$.

**Cavalieri’s* principle** If two solids of equal altitude, the section made by plane parallel to and the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

*Bonaventura Cavalieri, Italian mathematician (1598 – 1647)*

**center points** The *average*, the *median*, and the *mode* are so called “*center points*” that characterize a set of numerical data. You might use the average to find a center point that is midway between the extreme values of data (minimum and maximum). The median is a center point that is in the middle of all the data. That is, there are as many values less than the median as there are values greater than the median. The mode is a center point that represents the value or values that occur most frequently. The average/mean is useful for predicting future results when there are no extreme values in the data set. The median may be more useful than the average/mean when there are extreme values in the data set as it is not affected by extreme values. The mode is useful when the most common item, characteristic or value of a data set is required.

**center of a circle** The center of a circle is the given point that is a given distance from all points on the circle.

**center of an ellipse** The midpoint of the major axis.

**center of a hyperbola** The midpoint of the transverse axis.

**central angle in a circle** A central angle in a circle is an angle forms by two radii with the vertex at the center of a circle.

**central tendency** Numbers that describe where data are centered.

**centroid** 1. Centroid is the center of mass (or ‘center of gravity’ or barycenter) of an object/figure of uniform density. For example, the centroid of a triangle is the point which all the mass of a triangular plate seems to act. Geometrically speaking, centroid of a triangle is the point where the three medians
of the triangle intersect, and centroid is always inside the triangle. **N. B.** The centroid divides each median into two segments whose lengths are in the ratio $2:1$, with the longest one nearest the vertex.  

2. Centroid of a finite set is the point whose coordinates are the (arithmetic) mean values of the coordinates of the points of the set.

**characteristica universalis** *Characte**ristica universalis* is universal and formal language imagined by the famous German mathematician and (rational) philosopher **Gottfried Wilchelm Leibniz** (1646 – 1716) able to express mathematical, scientific and metaphysical concepts. Actually, he envisioned the creation of a universal language, i.e. *characteristica universalis*, ambitiously designed to serve as the vehicle for deduction and discovery in all fields of knowledge (Mikhail G. Katz and David M. Sherry: “Leibniz’s Laws of Continuity and Homogeneity”, Notices of the American Mathematical Society, December 2012, p. 1550). Leibniz explained that his goal was an alphabet of human thought, a universal symbolic language/characteristic for science, mathematics and metaphysics, but he never finish this his more than precious idea.

**Chebyshev’s theorem** This statistical theorem states that the *minimum percent* of data between ± $K$ standard deviations from the mean ($K > 1$) in any *distribution* can be found by the formula

$$
\text{Minimum percent} = 1 - \frac{1}{K^2}
$$

**chord** A line segment whose endpoints lie on a circle. Generally in plane geometry, a chord is the line segment joining two points on a curve.

**circle** (See “*conic sections*”) The set of all points in a plane that lie a given distance (*radius of a circle*) from a given point (*center of a circle*).

*“The power of the world always works in circles.”* (Lakota proverb)

**circular functions** The trigonometric functions (sine, cosine, tangent, cotangent, secant, and cosecant) of arc lengths, or real numbers, are also called circular functions.

**circumcenter of a triangle** The center of the circle that circumscribes the triangle, i.e. goes through all its vertices. In other words, it is the point of intersection of three lines drawn at right angles through the midpoints of each side.

**circumference** $C$ The distance around the circle, given by the formula $C = 2\pi r$.

**circumscribed circle** A circle that is drawn around the outside a triangle and contains all three vertices; a circle is circumscribed about a polygon if each vertex of the polygon lies on the circle.

**class** A group of data.

**clearing fractions** When working with an equation that contains fractions, multiplying both sides of the equation by the least common denominator (LCD) that will eliminate (clear) any denominators.
**closed-form expression/solution**  An expression/solution is said to be a closed-form expression/solution if it can be expressed analytically (i.e. in form of an equation/inequality) in terms of a finite number of certain elementary functions, elementary arithmetical operations (+, −, ·, ÷), nth roots, exponents and logarithms (which thus also include trigonometric functions and their inverses).

**closed interval**  The set of all real numbers \( x \) satisfying the double inequality \( a \leq x \leq b \) is called a closed interval with the points \( a \) and \( b \) as endpoints, and is denoted as \([a, b]\). In other words, this is an interval that includes both of its endpoints. For example, \([-3, 4]\). It means set of all real numbers between \(-3\) and \(4\), including endpoints \(-3\) and \(4\).

**closure**  The sum of any two natural numbers is an natural number. Therefore, the set of all natural numbers is said to be closed, or to satisfy the closure property, under the operation of addition. Strictly speaking, the closure is a property of some sets and binary operation. Closure means that the output of a binary operation is always within the set of the inputs.

**cluster sampling**  Cluster sampling in statistics is a sampling technique where the entire population is divided into groups or clusters, and a random sample of these clusters/groups are selected. We will use this technique when we cannot get a complete list of the members of population.

**coefficient**  In a term that is the product of a number and a variable(s), the number/numerical factor (in front of the variable) is the coefficient of the variable(s). For instance, in the term \(6x^2\), \(6\) is the coefficient of \(x^2\).

**cofunctions**  The function pairs sine and cosine, tangent and cotangent and secant and cosecant are called cofunctions.

**cofunction identities**  These identities state that cofunctions of the complementary angles are equal, i.e.

1. \(\cos (90^\circ − \theta) = \sin \theta\) or \(\sin (90^\circ − \theta) = \cos \theta\);
2. \(\tan (90^\circ − \theta) = \cot \theta\) or \(\cot (90^\circ − \theta) = \tan \theta\);
3. \(\sec (90^\circ − \theta) = \csc \theta\) or \(\csc (90^\circ − \theta) = \sec \theta\).

**collinear points**  The points are collinear if they lie on the same straight line.

**column matrix**  A matrix with exactly one column is a column matrix.

**columns of a matrix**  The vertical elements of a matrix. For example, the given matrix is an matrix with two rows and two columns. This type of matrix is a square matrix because the number of rows is equal to the number of columns.

\[
\begin{bmatrix}
1 & 3 \\
4 & 5
\end{bmatrix}
\]

**combinations**  Groupings of objects/things without regard to order. An combination means an distinct group or set of objects without regards to their arrangement. The number of combinations possible when \(r\) objects are selected from \(n\) objects is found by the combination formula
\[ \text{nCr} \text{ or } C(n,r) = \frac{n!}{r!(n-r)!}. \]

**Example** How many different samples of 5 apples can be drawn from a crate of 20 apples.

**Solution** If we want to choose 5 apples from a crate of 20 apples, then order is not important. That means this is an case of the combinations of 20 elements taken 5 at a time. The number of different ways the apples may be sampled is

\[
C(20, 5) = \frac{20!}{(20-5)!4!} = \frac{20!}{15!4!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15!4!} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} = 5 \cdot 19 \cdot 3 \cdot 17 = 4,845.
\]

**Combinatorics** Combinatorics is the branch of mathematics studying the *enumeration, combination, and permutation* of sets of elements and the mathematical relations that characterize their properties. Shortly speaking, combinatorics is the science of counting.

**combined variation** An variation (relationship) in which one variable varies directly and/or inversely, at the same time with more than one other variable. For example, equation \( y = \frac{kxz}{w^2} \), where \( k \) is a nonzero constant of variation, expresses combined variation. We can see that \( y \) varies jointly as \( x \) and \( z \), and inversely as the square of \( w \).

**commission** A percent of total sales paid to a salesperson.

**common difference** \( d \) In an arithmetic sequence \( a_1, a_2, a_3, \ldots, a_n \), the fixed number \( d \) that is added to each term to obtain the next term is the *common difference*.

**common factors** Numbers/expressions that are factors of the given numbers/expressions.

**common logarithm** A logarithm with base 10.

**common ratio** \( r \) In geometric sequence \( a_1, a_2, a_3, \ldots, a_n \), the fixed number \( r \) by which each term is multiplied to obtain the next term is the *common ratio*.

**commutative/abelian group** A group in which group operation satisfies commutative property.

**commutative law/property of addition/multiplication** The statement that two numbers can be added/multiplied in either order: the sum/product will be the same. For example, \( 2 + 5 = 5 + 2 = 7 \) or \( 2 \cdot 5 = 5 \cdot 2 = 10 \).

**Comparison Method** An algebraic method for solving systems of two equations in two variables/unknowns. In this method we will isolate either the \( x \) or \( y \) variables in both equation (1st step) and after that we will compare the other side of equations to derive the value of the other variable (2nd step). Now this derived value of the variable can be used by substituting it in one of the original variables to derive the value of the other variable (Final step).

**Example** Solve system of equations using the comparison method.

\[ 2x + y = 7 \quad (1) \]
3x + y = 10 \quad (2)

**Solution** We can see that the y variable is pretty easier to isolate.

(1) → y = 7 - 2x
(2) → y = 10 - 3x \quad (1^{st} \text{ step})
7 - 2x = 10 - 3x \quad (2^{nd} \text{ step}) → -2x + 3x = 10 - 7 → x = 3 →
(Final step) → y = 7 - 2 \cdot 3 = 7 - 6; \quad y = 1.

We will check the ordered pair (3, 1).

(1): 2x + y = 7
(2): 3x + y = 10
2(3) + 1 = 7 \quad (?)
3(3) + 1 = 10 \quad (?)
7 = 7 \text{ (True statement)} \quad 10 = 10 \text{ (True statement)}
Since (3, 1) checks, it is the solution.

**complementary angles/complements**  Two positive angles are complementary angles (or complements) if the sum of their measures is 90°.

**Example** Find the measure of complementary angles with measures $3x - 5$ and $6x - 40$ degrees.

**Solution** The sum of two complementary angles is 90°.

Therefore, $(3x - 5) + (6x - 40) = 90° \Rightarrow 9x - 45 = 90° \Rightarrow 9x = 135° \Rightarrow x = \frac{135°}{9} = 15°.$

The measures of the two complementary angles are $(3x - 5)^° = [3(15) - 5]^° = 40^°$ and $(6x - 40)^° = [6(15) - 40]^° = (90 - 40)^° = 50^°$.

**complement of an event** In probability, the set of all outcomes in a sample space that do not belong to an event $E$ is the complement of $E$, written $E'$.

**completing the square (procedure)** A method of adding a particular constant to an expression so that the resulting sum is a perfect square.

For example, $ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x + \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 \cdot a + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} \cdot a + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + \frac{4ac}{4a} = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}.$

**complex* fraction/rational expression** A quotient of two rational expressions/fractions or fraction where either the numerator or the denominator, or both, are fractions or mixed numbers. In other words, the term complex fraction describes actually a special kind of fraction in which the numerator, denominator, or both, are fractions.

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* **Simplifying a Complex Fraction:** 1. Write both the numerator and denominator as single fractions; 2. Change the complex fraction to a division problem; 3. Perform the indicated division.

**Example** Simplify.

\[
\frac{\frac{2}{5} + \frac{1}{4}}{\frac{1}{2} + \frac{1}{3}} = \frac{\frac{8}{20} + \frac{5}{20}}{\frac{3}{6} + \frac{2}{6}} = \frac{\frac{13}{20}}{\frac{5}{6}} \quad \Rightarrow \quad \frac{13}{20} \div \frac{5}{6} = \frac{13 \cdot 6}{20 \cdot 5} = \frac{13 \cdot 3}{10 \cdot 5} = 39.
\]
**complex number** The sum of a real number and an imaginary number constitutes what is known as a *complex number*. This term was introduced by Carl Friedrich Gauss in 1831. For example, $3 + 5i$ is an complex number, where $3$ is its real part, $5$ is imaginary part and $i$ is a mathematical symbol which is called *imaginary unit*. A set consisting of expressions of the form $z = a + bi$, where $i^2 + 1 = 0$, and $a$, and $b$ are real numbers, is a *set of complex numbers*.

**complex plane (Gauss’s coordinate plane)** The coordinate plane where the horizontal axis represents the real part of the complex number (real axis), while the vertical axis represents the imaginary part (imaginary axis).

**composite function (composition of functions)** If $f$ and $g$ are function, then the *composite function*, or *composition* of $f$ and $g$ is defined by $(f \circ g)(x) = f(g(x))$. The *domain* of $f \circ g$ is the set of all numbers $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f(x)$.

**Example** Given that $f(x) = \frac{2}{x+3}, x \neq -3$ and $g(x) = x + 1$, find $(f \circ g)(x)$ and its domain.

**Solution** The domain of $f$ is the set of all real numbers $x$, and $x \neq -3$, and the domain of $g$ (linear function) is set of all real numbers.

$(f \circ g)(x) = f(g(x)) = f(x+1) = \frac{2}{x+3}$

So, $x + 4 \neq 0 \Rightarrow x \neq -4$.

Therefore, the domain of $f \circ g$ is $(-\infty, -4) \cup (-4, \infty)$.

**composite number** A natural number, other than $1$, that is not prime. The integer $1$ is neither prime nor composite; it is called a *unit*.

**compound amount** In an investment paying compound interest, the compound amount is the balance after interest has been earned. The compound amount is sometimes called the *future value*.

**compound event** A combination of simple events.

**compound inequality** A statement in which two or more inequalities are joined by the word “and” or the word “or”.

**compound interest** In compound interest*, interest is paid both on the principal and previously earned interest. In other words, this is interest computed on the sum of an original principal and the interest
previously accrued by that principal.

*If \( P \) dollars are deposited in an account paying an annual rate of interest \( r \) compounded or paid \( n \) times per year, then after \( t \) years the account will contain \( A \) dollars, where \( A = P \left( 1 + \frac{r}{n} \right)^{tn} \).

**concave polygon** A polygon that is not convex.

**concavity** The turning or bending behavior of the function defines the concavity of the graph of the function. More precisely, a function is **concave upward** on the given interval \((a, b)\) if the graph of the function lies above its tangent line at each point of \((a, b)\). A function is **concave downward** on \((a, b)\) if the graph of the function lies below its tangent line at each point of \((a, b)\).

*Test for concavity* Let \( f \) be a function with derivatives \( f' \) and \( f'' \) existing at all points in the given interval \((a, b)\). Then \( f \) is **concave upward** on \((a, b)\) if \( f''(x) > 0 \) for all \( x \) in \((a, b)\), and **concave downward** on \((a, b)\) if \( f''(x) < 0 \) for all \( x \) in \((a, b)\).

**Example** Determine the concavity of \( f(x) = 2x^3 + 6x^2 - 5x + 3 \).

**Solution** We can see that domain of \( f(x) \) is the set of all real numbers, because it is a polynomial function.

\[
f'(x) = 6x^2 + 12x - 5,
\]
\[
f''(x) = 12x + 12,
\]
\[
f''(x) = 0 \Rightarrow 12x + 12 = 0 \Rightarrow 12x = -12 \Rightarrow x = -1 \Rightarrow f(-1) = -2 + 6 + 5 + 3 = 12.
\]
Testing intervals to the left and to the right of \( x = -1 \).

For \( f''(x) = 12x + 12 \), we find that
\[
f''(-2) = -24 + 12 = -12 < 0 \Rightarrow f''(x) < 0, \forall x \in (-\infty, -1) \text{ and } f''(0) = 12 > 0 \Rightarrow f''(x) > 0, \forall x \in (-1, \infty).
\]
Therefore, \( f(x) \) is **concave downward** on \((-\infty, -1)\) and **concave upward** on \((-1, \infty)\), and the function \( f(x) \) has a **point of inflection** at \((-1, 12)\).

**conclusion** The phrase following the word *then* in a conditional statement; the final statement of an argument.

**conditional equation** An equation that is satisfied only by some numbers. For example, equation \(3x = 12\) is a conditional equation, and its **solution set** is \(\{4\}\). That means this equation is true only for \(4\) and false for other numbers.

**conditional probability** The probability of event \(E_2\) occurring, given that an event \(E_1\) has happened (or will happen; the time relationship does not matter), called a conditional probability and is written \(P(E_1 \mid E_2)\). For any two events, \(E_1\) and \(E_2\), the conditional probability, \(P(E_1 \mid E_2)\), is determined by formula

\[
P(E_1 \mid E_2) = \frac{n(E_1 \text{ and } E_2)}{n(E_1)},
\]
where \(n(E_1 \text{ and } E_2)\) represents the number of sample points/outcomes common to both event 1 and event 2, and \(n(E_1)\) is the number of sample points/outcomes in event \(E_1\).
conditional statement  A statement that can be written in the form "If $p$, then $q$", where $p$ is the hypothesis and $q$ is the conclusion.

cone  A 3-dimensional figure whose base is a circle and whose height is the perpendicular distance from the base to the vertex.

congruence The relationship between figures having the same shape and the same size.

congruent angles Two angles that have the same measure.

congruent polygons Two polygons in which each pair of corresponding angles is congruent and each pair of corresponding sides is congruent.

conic sections (conics)*Conics** are curves (circle, parabola, ellipse and hyperbola) of intersection of various planes with lateral surface of a cone. A conical surface is imagined as extending in both directions from the vertex without limit. Or more precisely, a conic section is curve obtained by intersecting a right circular conical surface with a plane. All conics can be characterized by one general definition. A conic is the set of all points $P(x, y)$ in a plane such that the ratio of the distance from $P$ to a fixed point and the distance from $P$ to a fixed line is constant. The constant ratio is called the eccentricity of the conic, written $e$ (Circle: $e = 0$; Parabola: $e = 1$; Ellipse: $0 < e < 1$; Hyperbola: $e > 1$).

*Horizontal parabola (axis of symmetry parallel to the x-axis), opens right ($p>0$) or opens left ($p<0$)

$$(y - k)^2 = 4p(x - h), \text{ Vertex: V}(h, k), \text{ Focus: F}(h + p, k), \text{ Directrix: } x = h - p; \text{ Axis: } y = k$$

Vertical parabola (axis of symmetry parallel to the y-axis), opens up ($p>0$) or opens down ($p<0$)

$$(x - h)^2 = 4p(y - k), \text{ Vertex: V}(h, k), \text{ Focus: F}(h, k + p), \text{ Directrix: } y = k - p; \text{ Axis: } x = h$$

If the equation of the vertical parabola is given in the explicit form $f(x) = ax^2 + bx + c$, then we can find its vertex using vertex formula $V[-\frac{b}{2a}; f(-\frac{b}{2a})]$, i.e. $h = -\frac{b}{2a}; k = f(-\frac{b}{2a}) = f(h)$

Eccentricity $e$ of every parabola is 1.

Example 1  Write an equation for parabola with vertex at (-2, 1), and focus at (-2, -3).

Solution  Since the focus is below the vertex, the axis of symmetry is vertical and the parabola opens downward (vertical parabola). The distance between the vertex and the focus is $d(V, F) = 1 - (-3) = 4$. Since the parabola opens downward, the parameter $p$ will be negative. So, $p = -4$.

The equation of the given parabola will have the form $(x - h)^2 = 4p(y - k)$.

Substitute $p = -4, h = -2,$ and $k = 1$ to find required equation. $[x - (-2)]^2 = 4(-4)(y - 1)$ or $(x + 2)^2 = -16(y - 1)$.

Horizontal ellipse (major axis parallel to the x-axis) with center at $(h, k)$
\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \text{ Foci: (h + c, k) & (h - c, k), Vertices: (h + a, k) & (h - a, k); } a^2 = b^2 + c^2, \quad a > b \quad \& \quad a > c
\]

**Vertical ellipse (major axis parallel to the y-axis) with center at (h, k)**

\[
\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \text{ Foci: (h, k + c) & (h, k - c), Vertices: (h, k + a) & (h, k - a)}
\]

**Eccentricity of ellipse:** \[ e = \frac{c}{a} \quad (0 < e < 1) \]

**Example 2**  Write an equation for ellipse with center at (-2, 7); major axis vertical, with length 10, c = 2

**Solution** Since the center is (-2, 7) and major axis is vertical (vertical ellipse), the equation has the form \[
\frac{(x+2)^2}{b^2} + \frac{(y-7)^2}{a^2} = 1. \]
The major axis has length 10, so \(a = 5\). Using equation \(c^2 = a^2 - b^2\) we will find \(b^2\).

So, \(4 = 25 - b^2\) \(\Rightarrow\) \(b^2 = 21\).

Therefore, the equation of the ellipse is \[
\frac{(x+2)^2}{21} + \frac{(y-7)^2}{25} = 1.
\]

**Area of a ellipse**  The area of a ellipse is given by formula:

\[ A = \pi ab \]

**Horizontal hyperbola** (transverse axis parallel to the x-axis) with center at (h, k); \(c^2 = a^2 + b^2, \quad a < c\)

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \text{ Foci: (h ± c, k), Vertices: (h ± a, k), Asymptotes: } y - k = \pm \frac{b}{a}(x - h)
\]

**Vertical hyperbola** (transverse axis parallel to the y-axis) with center at (h, k)

\[
\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1, \text{ Foci: (h, k ± c), Vertices: (h, k ± a), Asymptotes: } y - k = \pm \frac{a}{b}(x - h)
\]

**Eccentricity of hyperbola:** \[ e = \frac{c}{a} \quad (e > 1) \]

**Example 3**  Find an equation of the hyperbola with center at (9, -7), focus at (9, -17) and vertex at (9, -13).

**Solution** Since the center, focus, and vertex are on a vertical transverse axis (because their first coordinate is the same), the equation of hyperbola has the form \[
\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \quad \text{(vertical hyperbola)}
\]
The distance from the center to the vertex being 6 implies \(a = 6\). The focus (9, -17), is 10 units from the center, so \(c = 10\). Given that \(c^2 = a^2 + b^2\), we have \(10^2 = 6^2 + b^2 \Rightarrow b^2 = 64\).

Therefore, the equation of the hyperbola is \[
\frac{(y+7)^2}{36} - \frac{(x-9)^2}{64} = 1
\]

**Circle** with center at (h, k) and radius r

\[(x - h)^2 + (y - k)^2 = r^2.\]

**Example 4**  Find the equation of a circle with center at C(-4, 3), passing through the point P(5, 8). Write it in center-radius form.

**Solution** The radius, by the definition, is the distance from the center C(-4, 3) to the point P(5, 8). So,

\[ r = d(C, P) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[5 - (-4)]^2 + (8 - 3)^2} = \sqrt{9^2 + 5^2} = \sqrt{106} \Rightarrow (x + 4)^2 + (y - 3)^2 = 106.\]

**Conic sections are among the oldest curves, and one of the oldest mathematical challenging subjects**
studied systematically and thoroughly. The conic seems to have been discovered by Menaechmus, ancient Greek mathematician and geometer (ca. 375 – 325 BC), tutor to Alexander the Great (together with Aristotle). They were conceived in an attempt to solve the three famous ancient construction problems of trisecting the angle, doubling the cube, and squaring the circle (These ancient problems in elementary geometry lingered until early 19th century when it was shown that it is impossible to solve them with the help of only a straightedge and compass). First scientific synthesis about the nature of the conic sections gave Roger (Rudjer) J. Boscovich (1711-1787), mathematician, physicist, philosopher, astronomer (one crater on the Moon has his name), geodesist, engineer, architect, diplomatist, priest and poet in the true sense of these words. He published this original theory of conic sections in 1754 in the third and final volume of his textbook “Elementa universae nathesoes” as professor of mathematics at the ‘Collegium Romanum’ in Rome, Italy. He was born at Ragusa in Republic of Ragusa (today Dubrovnik, Croatia) of Serb and Italian parentage. The first general mathematical theory of atomism and the first fundamental steps in theory of relativity was his work (It appeared in 1758). Rudjer Boscovich published about hundred books and papers. In this his famous work “Theoria philosophiae naturalis redacta ad unicum legem virium in natura existentium” (Theory of natural philosophy derived to the single law of forces which exist in nature), he made the great attempt in this work to understand the structure of the universe in terms of a simple idea, and he developed the first coherent description of atomic theory. He held that bodies could not be composed of continuous matter, but countless “point-like structures”. The fourth edition of this Boscovich’s famous book was published in 1966 in United States of America. He was well known especially for his ideas about ‘atoms’ and the forces between them. According to Boscovich atoms are structureless points which exhibit repelling and attracting forces on each other depending of distance. This person of the renaissance’ versatility and one of the great intellectual figures all ages was, according the English physicist, mathematician, and inventor John Henry Poynting (1852-1914), “amongst the boldest minds humanity has produced”.

conjecture Proposition before it has been proved or disproved or a statement that is believed to be true. For example, Goldbach’s Conjecture states that every even natural number, except 2, is the sum of two prime numbers.

conjugates The expressions $a - b$ and $a + b$ are called conjugates.
conjugate of a complex number  The conjugate of a complex number $a + bi$ is $a - bi$.

Conjugate Zeros Theorem  If $f(x)$ defines a polynomial function having only real coefficient and if $z = a + bi$ is a zero of $f(x)$, then $Z = a - bi$ (conjugate of $z$) is also a zero of $f(x)$.

Example  Find a polynomial function of least degree having only real coefficients with zeros as given: $-1$ and $4 - 2i$.

Solution  By the conjugate zeros theorem, $4 + 2i$ must also be a zero.

$$f(x) = (x + 1)(x - (4 - 2i))(x - (4 + 2i)) = (x + 1)(x - 4 + 2i)(x - 4 - 2i) =$$

$$= (x + 1)((x - 4) + 2i)((x - 4) - 2i) = (x + 1)((x - 4)^2 - 4i^2) = (x + 1)(x^2 - 8x + 16 + 4) =$$

$$= (x + 1)(x^2 - 8x + 20) = x^3 - 7x^2 + 12x + 20.$$ 

N. B.  If $x_1, x_2, \ldots, x_n$ are zeros of an polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, then we can express $f(x)$ in factored form as follows: $f(x) = a_n(x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n)$.

conjunction  A sentence in which two or more sentences are joined by the word and to make a compound sentence.

consecutive even/odd integers  Even/odd integers that are two units apart. For instance, if the first even/odd integer is $x$, then the next even/odd integer is $x + 2$.

consecutive integers  Two integers that are one unit apart. For instance, if the first integer is $x$, then the next integer is $x + 1$.

consequent  See ratio.

consistent system of equations  A system of equations that has at least one solution.

constant*  A specific number that never changes in time. In other words, algebraically speaking, this is a monomial without a variable. For example, in $5x^2 - 3x + 7$ the number 7 is the constant term.

*The speed of light in vacuum $c$, the gravitational constant $G$, and Planck’s constant $h$ are most important constants in universe (See Fundamental Physical Constants & Planck-Einstein equation).

constant function  A function $y = f(x)$ is constant on an interval $I$ if, for every $x_1$ and $x_2$ in $I$, $f(x_1) = f(x_2)$.

constant of variation/proportionality  The constant, $k$, in an equation of direct or inverse variation; also called variation constant.

constrains  In linear programming, the inequalities that represent restrictions (limitations) on a particular situation are called the constrains.

consumers’ surplus  If $D(q)$ is a demand function with equilibrium price $p_0$ and equilibrium demand $q_0$, then

$$Consumers’ \text{ surplus} = \int_0^{q_0} [D(q) - p_0] \, dq.$$  

N. B. $p_0 = D(q_0)$

Example  Find the consumers’ surplus if the demand function for grass seed is given by equation
\[ D(q) = \frac{200}{(3q+1)^2}, \]

assuming supply and demand are in equilibrium at \( q_0 = 3 \).

**Solution** The equilibrium price \( p_0 \) for the given equilibrium demand \( q_0 \) will be
\[ p_0 = D(q_0) = D(3) = \frac{200}{(3\cdot3+1)^2} = \frac{200}{100} = 2. \]

Therefore, for the consumers’ surplus we will get
\[
\int_0^3 \left[ \frac{200}{(3q+1)^2} - 2 \right] dq = 1 \int_0^3 \frac{200}{(3q+1)^2} dq - 2 \int_0^3 dq =
\]
\{ Let \( u = 3q + 1 \), so that \( du = 3dq \) and \( dq = \frac{1}{3}du \); \( q = 0 \Rightarrow u = 1, q = 3 \Rightarrow u = 10 \} \]
\[ = \frac{1}{3} \int_1^{10} \frac{200}{u^2} \, du - 2 \int_0^3 dq = \frac{200}{3} \int_1^{10} u^{-2} \, du - 2 \int_0^3 dq = \frac{200}{3} \cdot \frac{u^{-1}}{-1} \bigg|_1^{10} - 2q \bigg|_0^3 = -\frac{200}{3u} \frac{10 - 6}{10 - 1} - 6 =
\]
\[ = \frac{200}{30} + \frac{200}{3} - 6 = -\frac{200}{30} + \frac{200}{30} = \frac{1620}{30} = 54. \]

**Continuity of a Function at a Point**  1. A function \( f(x) \) is called **continuous** at a point \( a \) if the limit of the function \( f(x) \) at the point \( a \) exists and is equal to the value of the function for this point:
\[
\lim_{x \to a} f(x) = f(a). \]

In other words, continuity of a function \( f(x) \) at a point \( a \) means simultaneous feasibility of the following conditions:

(1) the function \( f(x) \) must be defined at the point \( a \);
(2) the function \( f(x) \) must have a limit at the point \( a \);
(3) the limit of the function \( f(x) \) at the point \( a \) coincides with the value of the function at this point.

2. It is possible to give another definition of continuity of a function using the notions of the increment in the argument (input), \( \Delta x \), and function (output), \( \Delta f \). Namely, a function \( f(x) \), \( x \in (a, b) \) is called continuous at a point \( x_0 \in (a, b) \) if
\[
\lim_{\Delta x \to 0} \Delta f = 0,
\]
where \( \Delta x = x - x_0 \), the increment in the argument \( x \), and \( \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) \), the increment in the function \( f(x) \).

In an informal definition we can say that a function is continuous over an interval of its domain if its hand-drawn graph over that interval can be sketched without lifting the pencil from the paper.

**continuous compounding** Continuous compounding of money involves the computation of interest as the frequency of compounding approaches infinity, leading to the formula \( A = Pe^{rt} \), where \( A \) is the compounded amount, \( P \) is deposited money at a rate of interest \( r \) compounded continuously for \( t \) years.

**contraction** A dilatation in which the preimage is reduced in size.

**contradiction** 1. An equation that has no solution (solution set is empty). For example, \( x + 3 = x + 5 \) is an contradiction, because we will get an false statement \( 3 = 5 \) (or \( 0 = 2 \)).

2. Contradiction in logic has the following form: \( p & \sim p \). That is, a contradiction asserts that a statement and its negation are both true.

...............
*Blaise Pascal* (1623-1662): “Contradiction is not a sign of falsity, nor the lack of contradiction a sign of truth.”

**contrapositive of a conditional** The statement formed by interchanging the hypothesis and conclusion of a conditional statement and negating each part. In other words, the contrapositive of a statement \( p \rightarrow q \) is the statement \( \neg q \rightarrow \neg p \). If conditional is true, its contrapositive must also be true.

**convergent sequence** An infinite sequence is convergent if its terms get closer to some real number (limit of the sequence).

**convergent infinite geometric sequence** The infinite geometric sequence \( a_1, a_2, a_3, \ldots \) is convergent if \( |r| < 1 \), and its sum is given by \( S = \frac{a_1}{1-r} \).

**converse of a conditional** The converse of a conditional statement is found by interchanging the hypothesis and conclusion. The converse of a statement \( p \rightarrow q \) is the statement \( q \rightarrow p \).

The converse of a true conditional statement may or not be true.

**Example**

**Conditional:** If the product of two numbers is 0, then at least one of the numbers must be 0.

**Converse:** If at least one of the numbers is 0, then the product of the two numbers is 0.

We can see for this case that the converse is also true statement, but sometimes the original conditional is true and converse is false.

**conversion of angular measure** To convert degree measure to radian measure of an angle we will use degree/radian relationship \( 360^\circ = 2\pi \) **radians** or \( 180^\circ = \pi \) **radians**, i.e.

1) \( 1^\circ = \frac{\pi}{180^\circ} \) **radian**

2) \( 1 \) **radian** = \( \frac{180^\circ}{\pi} \)

**convex polygon** If all diagonals of a polygon lie inside the figure, the polygon is termed **convex**. If not, the polygon is non-convex or **concave**.

**coordinate** A number that corresponds to a particular point on a number line.

**coordinate plane (xy-plane)** A grid formed by two (perpendicular) number lines (axes) that intersect (at right angle) at point known as a **origin**. Each point on a plane is the graph of a ordered number pair. The first coordinate of an ordered pair is always graphed in horizontal direction (x-axis) and the second coordinate is always graphed in vertical direction (y-axis).

**coordinate system** (on a number line, in the x-y plane, in the x-y-z space,...) The correspondence between points on a line (in the x-y plane, or in the x-y-z space) and the real numbers is called a coordinate system (in one, two or three dimensions).

**coordinates of a point on the plane** The **abscissa** (x-value) and **ordinate** (y-value) of a point, used to locate the point on the coordinate plane.
coplanar Lying in the same plane.

corresponding angles Two nonadjacent angles, one interior and one exterior, that lie on the same side of a transversal.

corollary of a theorem A theorem that follows directly from another theorem and that can easily be proved from that theorem.

cosecant In a right triangle, the ratio of the length of the hypotenuse to the length of the opposite side of the given acute angle.

cosine In a right triangle, the ratio of the length of the side adjacent to an acute angle to the length of the hypotenuse.

cost function A function which describes the cost of manufacturing some products. The cost of manufacturing an item commonly consists of two parts. The first part is a fixed cost for designing the product, setting up a factory, and so on. The second part is a cost per item for labor, materials, packing, shipping, and so on. If a cost function is a linear function of the slope-intercept form \( C(x) = mx + b \), then the \( m \) represents the marginal cost per item and \( b \) is the fixed cost.

cotangent In a right triangle, ratio of the length of the side adjacent to an acute angle to the length of the opposite side of the given acute angle.

coterminal angles Two angles that have the same initial side and the terminal side, but different amounts of rotation, are called coterminal angles. The measures of coterminal angles differ by a multiple of 360° (or \( 2\pi \) radians).

Example Find the angle of least positive measure (not equal to the given measure) coterminal with angle -98°.

Solution Coterminal angles have the same initial side and the same terminal side. Their measure differ by multiple of 360° (i.e., 360° or 2 \times 360° or 3 \times 360° or \ldots n \times 360°).

For this case we will have 360° + (-98°) = 262°.

counterexample An example that proves that a statement (initial example), often a conjecture, is false. In other words, the counterexample always shows that conjecture is false.

counting principle If one event can happen in \( n \) ways and a second event can happen in \( m \) ways, the two events can occur together in \( m \cdot n \) ways.

CPCTC Abbreviation for “corresponding parts of congruent triangles are congruent.”

Cramer’s rule Cramer’s rule is a specific method of using determinants to solve a system of linear equations.

critical numbers The critical numbers for a function \( y = f(x) \) are those numbers \( c \) (x-values) in the domain of \( f(x) \) for which \( f'(c) = 0 \) or \( f'(c) \) does not exist.
critical pedagogy  Critical pedagogue Ira Shor (1945) defines critical pedagogy as: “Habits of thought, reading, writing, and speaking which go beneath surface meaning, first impressions, dominant myths, official pronouncements, traditional cliches, received wisdom, and mere opinions, to understand the deep meaning, root causes, social context, ideology, and personal consequences of any action, event, object, process, organization, experience, text, subject matter, policy, mass media, or discourse.”

critical point  A point whose x-coordinate is the critical number  c, and whose y-coordinate is \( f(c) \). In addition, \( f(c) \) is so called stationary value.

critical teaching  The critical teaching  is an function of the critical thinking with special relationships with Socratic teaching.

critical thinking  Critical thinking* is the art of analyzing and evaluating thinking with a view to improving it, using universal intellectual standards**.

* Gautama Buddha (563 BC – 483 BC): “We are what we think.”

**Universal Intellectual Standards:
1) Clarity; 2) Accuracy; 3) Precision;
4) Relevance; 5) Depth; 6) Breadth;
7) Logic; 8) Significance; 9) Fairness

Cross Product of Two Vectors in Space  See Vector Product.

cube  A 3–dimensional figure that contains six square faces. At each vertex, all sides meet at right angles.

cube root  The number \( a \) is called cube root of \( b \) if \( a^3 = b \) or \( a = \sqrt[3]{b} \).

cubic equation (or equation of 3rd degree)  An equation of the form \( ax^3 + bx^2 + cx + d = 0 \). The scholar Omar Khayyam* (1048 – 1131) of Nishapur, brilliant Persian (Iranian) mathematician, astronomer and the famous classic of Persian and Tajik poetry (especially his collection of quatrains The Rubaiyat) made a systematic study of equations of third degree. He developed a method by which it is possible geometrically to find the number of real roots of a cubic equation (he himself was only interested in positive roots). For example, he found a positive root of equation \( x^3 + 200x = 20x^2 + 200 \) by considering the intersection of a rectangular hyperbola and a circle (!). Original problem actually was about solving a right triangle having the property that hypotenuse equals the sum of one leg and the altitude on the hypotenuse. This master of accuracy also measured the length of the year as 365.24219858156 days. Just an unbelievable degree of accuracy because we know today that the length of the year is changing in the sixth decimal place over a person’s life time. For example, the length of the year at the end of the 19th century was 365.242196 days, while today it is 365.242190 days.

*Some Khayyam’s quotations
1) A hair divides what is false and true. 2) Be happy for this moment. This moment is your life. 3) When I want to understand what is happening today or try to decide what will happen tomorrow, I look back.

**curvature** The rate at which the unit tangent vector \( T \) turns per unit of length along the curve is called the *curvature*. The symbol for the curvature function is the Greek letter \( \kappa \) (“kappa”). Actually, the curvature measures the failure of a curve to be a straight line. If \( T \) is the unit tangent vector of a smooth curve, the *curvature* function of the curve is 
\[
\kappa = \left| \frac{dT}{ds} \right|. 
\]

**cycloid** A cycloid is a curve that represents the path traced by a fixed point on the circumference of a circle rolling along a line.

**cylinder** A 3-dimensional figure that has both circular base and circular top, and whose height is the perpendicular distance from the top to the bottom.

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**D**

**damping** A decreasing of the amplitude of an electrical or mechanical wave.

**data** The facts, or numbers, that describe something or information in the form of numbers or numbers collected in an experiment.

**day** The 24 hour period from the midnight to the next midnight. It is how long the Earth takes to spin on its axis.

**decagon** The polygon with 10 sides.

**decimals** Digits to the right of the ones place.

**decimal fraction** A fraction whose denominator is a power of 10. For example, \( \frac{7}{10} \), \( \frac{11}{100} \), and \( \frac{131}{1000} \) are decimal fractions.

**decimal number** A decimal number is another way of representing a fraction/rational number. The decimals/decimal numbers are read as whole numbers followed by the name of the rightmost place. For example, the decimal number 2.04583 we will read: “Two and four thousand five hundred eighty-three hundred thousandths.”

**decimal point** A point that shows where the whole number ends and, at the same time, where the decimals begin.
**decimal system of numbers (Hindu-Arabic number system)** A number system that uses a notation in which each number is expressed in base 10 by using one of the first nine natural numbers or 0 in each place and letting each place value be power of 10. Famous Italian mathematician **Leonardo Pisano Fibonacci** (c. 1170-c. 1250) was one of the first people to introduce the Hindu-Arabic decimal number system into Europe – just place-valued decimal system we use today – based on ten digits with its decimal point and a symbol for zero: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

*In many modern languages, the names of all numbers up to a million are made up of only 37 words denoting the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000. The underlying element of all these 37 word formations is the number 10. Our 10 fingers on our two hands are best explanation for this special and exceptional role of 10.*

**decreasing function** A function \( f \) is decreasing on an interval \( A \) if, whenever \( x_1 < x_2 \) in \( A \), \( f(x_1) > f(x_2) \).

**deductive reasoning** The process of reasoning to a specific conclusion from a general statement. Deductive reasoning is actually the logical application of general principles to predict a specific result. In science deductive reasoning is used to test the validity of general ideas.

**deferral** An deferral is a postponement of the recognition of an expense already paid or of a revenue already received.

**definite integral** Let there be given a function \( f(x) \), \( x \in [a, b] \). If the limit
\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]
exists and is independent of the choice of points \( x_i \), then the function \( f(x) \), is said to be integrable on the interval \([a, b]\), and the limit is called the definite integral of the function \( f(x) \) over the interval \([a, b]\) and is denoted by
\[
\int_{a}^{b} f(x) dx.
\]
This designation is read “The integral from \( a \) to \( b \) of the function \( f(x) \) with respect to \( dx \)” or, briefer, “The integral from \( a \) to \( b \) of \( f(x) dx \).” The numbers \( a \) and \( b \) are called (respectively) the lower and upper limits* of integration.

*It is unfortunate that the word “limit” is used in this connection. The limits of integration have nothing to do with the limits of a function.*

**degree of an algebraic equation** After transposing all terms of an algebraic equation to one side and collecting like terms; if the equation contains only one unknown, then degree of the equation is the greatest exponent on the unknown. If the equation contains several unknowns, then for each term of equation it is necessary to form the sum of the exponents of all unknown. The largest sum is termed the degree of the equation.

**Example 1** The equation \( 5x^3 + 12x^2 - 11x = 5x^3 - 25 \) is second-degree equation since we get
12x^2 - 11x + 25 = 0 when all terms are transposed to the left side, then the equation is in so called *standard form*.

**Example 2** The equation $3a^3x^4 - 2bx^3y^4 + 5axy^5 - 7 = 0$ is an equation of the seventh degree since the sum of the exponents of the unknowns $x$ and $y$ comes to 4 for the first, 7 for the second term, 6 for the third and zero for the fourth; the largest sum is 7.

**degree* measure of angles** The most common unit for measuring angles is the degree. The symbol is $^\circ$. One degree ($1^\circ$) represents $\frac{1}{360}$ of one whole rotation.

*Degree measure was developed by the Assyrians and Babylonians, 4000 years ago. They were the first to subdivide the circumference of a circle into 360 parts (degrees). There are numerous hypotheses as to how this occurred but there is no firm proof for any of them.

**degree of a polynomial** The greatest degree of any term in a polynomial is the degree of the polynomial. For example, the polynomial $3x^4 - 2x^3 + x^2 - 7x + 5$ has degree 4.

**demand function** $D(q)$ The function of $q$, defined by $p = D(q)$, relates the number of units $q$ of an item that consumers are willing to purchase to the price $p$. But remember, it is really *price* that determines how much consumers demand and producers supply. In other words, if *price decreases*, then *demand increases* and the *supply decreases*, and inversely.

**De Moiver's* Theorem/Formula** De Moivre’s theorem/formula states that for any complex number $r(cos \theta + isin \theta)$ and any integer $n$ holds that $[(r(cos \theta + isin \theta))^n = r^n(cos n\theta + isin n\theta)$.

This formula actually links complex numbers and trigonometry.

*Abraham De Moivre (1667 – 1754), French mathematician.*

**denominator** The divisor in a quotient expressed in fraction form. For example, 3 is the denominator of $\frac{7x}{3}$. It indicates the number of parts into which one whole is divided.

**density property for real numbers** The density property for real numbers states that between any two real numbers, no matter how close, there exists another real number. In other words, set of all real numbers $R$ (with its subsets) is a dense set.

**dense set** A set of numbers is said to be a dense set if between any two distinct members of the set there exists a third distinct member of the set.

**dependent events** Two or more separate events that affect the happening of one another.

**dependent system of equations** A system with an infinite number of solutions.
dependent variable If the value of the variable $y$ depends on the value of the variable $x$, then $y$ is called the dependent variable.

depreciation How much the value of an item is lowered after a certain time. In other words, when a piece of equipment is used, its value decreases every year. This is called *depreciation*.

derivative The derivative* of the function $y = f(x)$ with respect to the variable/argument $x$ is the function $f'(x)$, defined by the limit $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, provided the limit exists. The operation of finding the derivative of a given function is called *differentiation*. A function having a derivative at some point is called *differentiable* at this point. The derivative of a function at a point actually measures the rate at which the function’s value changes as the function’s argument changes. In addition to, we can interpret derivative as an instantaneous rate (velocity) of change of the function (physical meaning) or slope of the tangent line of the graph of the function at given point (geometrical meaning).

*The derivative exists at some point when a function $f(x)$ satisfies conditions: (a) $f(x)$ is continuous; (b) $f(x)$ is smooth; (c) $f(x)$ does not have a vertical tangent line at this point.*

**Derivative of an Inverse Function** If a function $y = f(x)$, $x \in (a, b)$, and its inverse $x = f^{-1}(y)$ have derivatives, and $\frac{df}{dx} \neq 0$, then

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}} \quad \text{or} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

**Derivatives of simplest elementary functions & Basic differentiation formulas**

1. Constant: $(c)' = 0$; 2. Power: $(x^n)' = nx^{n-1}$;

3. Sum/Difference: $(f \pm g)' = f' \pm g'$; 4. Product: $(fg)' = f'g + fg'$;

5. Quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$; 6. Composite function – Chain Rule: $(f(g(x)))' = f'(g) \cdot g'(x)$;

7. Exponential and logarithmic function: (a) $(e^x)' = e^x$; b) $(a^x)' = a^x \ln a$;

8. Trigonometric functions: $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$;

$(\tan x)' = \frac{1}{\cos^2 x}$ or $\sec^2 x$; $(\cot x)' = -\frac{1}{\sin^2 x}$ or $-\csc^2 x$;
(9) Inverse trigonometric function: 
\[
\begin{align*}
\arcsin x' &= \frac{1}{\sqrt{1-x^2}}; & \arccos x' &= -\frac{1}{\sqrt{1-x^2}}; \\
\arctan x' &= \frac{1}{1+x^2}; & \arccot x' &= -\frac{1}{1+x^2};
\end{align*}
\]

(10) Function represented parametrically: If \( x = x(t) \) and \( y = y(t) \) then
\[
y'(x) = \frac{y'(t)}{x'(t)} = \frac{y'_t}{x'_t}.
\]

Example 1  The total number of bacteria (in millions) present in a culture is given by
\[
N(t) = 2t \left( 5t + 9 \right)^{\frac{1}{2}} + 12,
\]
where \( t \) represents time (in hours) after the beginning of an experiment. Find the rate of change of the population of bacteria \( (N'(t)) \) with respect to time for the following numbers of hours
\[
(a) \quad 0 \quad (b) \quad \frac{7}{5} \quad (c) \quad 8.
\]

Solution  We will use here
\[
\begin{align*}
(1) \quad \text{Product Rule:} \quad (fg)' &= f'g + fg' \; ; \\
(2) \quad \text{Power Rule \& Chain Rule:} \quad (f(x)^n)' &= n \cdot f^n \cdot f' \; .
\end{align*}
\]

\[
N'(t) = 2 \left( 5t + 9 \right)^{\frac{1}{2}} + 2t \left( \frac{1}{2} \right) (5t + 9)^{-\frac{1}{2}} (5t + 9)' = 2(5t + 9)^{\frac{1}{2}} + t(5t + 9)^{-\frac{1}{2}} \cdot 5
\]
\[
N'(t) = 2(5t + 9)^{\frac{1}{2}} + \frac{5t}{(5t+9)^{\frac{1}{2}}} = \frac{2(5t+9) + 5t}{(5t+9)^{\frac{1}{2}}} \quad \text{or} \quad N'(t) = \frac{15t+18}{(5t+9)^{\frac{1}{2}}}.
\]

\[
(a) \quad N'(0) = \frac{18}{\sqrt{9}} = \frac{18}{3} = 6;
\]

\[
(b) \quad N'(7/5) = \frac{15 \cdot \frac{7}{5} + 18}{(5 \cdot \frac{7}{5} + 9)^{\frac{1}{2}}} = 39/4 = 9.
\]

\[
(c) \quad (c) \quad N'(8) = \frac{120+18}{\sqrt{49}} = 138/7 \approx 19.71.
\]

Example 2  Find the derivative of the function \( h(x) = [f(x)]^{g(x)}. \)

Solution  We can see, this is a special type of the composite exponential function.

1\textsuperscript{st} step:  We will find logarithm of \( h(x). \)
\[
h(x) = \left[ f(x) \right]^{g(x)} / \ln
\]
\[
\ln h(x) = \ln \left[ f(x)^{g(x)} \right] \rightarrow \ln h(x) = g(x) \ln f(x) \quad \text{(1)} \quad \text{(Power property of logarithms: } \ln a^n = n \ln a).\]

2\textsuperscript{nd} step:  We will differentiate equation (1).
\[
\ln h(x) = g(x) \ln f(x) \left( \frac{d}{dx} \right) \rightarrow \frac{1}{h(x)} h'(x) = g'(x) \ln f(x) + g(x) \cdot \frac{1}{f(x)} f'(x) \cdot h(x) \rightarrow
\]
\[
h'(x) = h(x) \left[ g'(x) \ln f(x) + \frac{g(x)f'(x)}{f(x)} \right] \quad \text{or}
\]
\[ h'(x) = f(x)g(x)\left(\frac{f'(x)g(x)\ln f(x) + g(x)f'(x)}{f(x)}\right) \quad \text{or} \]

\[ h'(x) = [f(x)]^{g(x)-1}[f(x)g'(x)\ln f(x) + g(x)f'(x)]. \]

**Example 3** Make sure that function represented parametrically by equations \( x = \frac{1+t}{t^3}, \quad y = \frac{3}{2t^2} + \frac{2}{t} \)
satisfies the relationship \( xy'^3 = 1 + y' \), where \( y' = \frac{dy}{dx} \), i.e. differentiation with respect to \( x \).

**Solution**

1\(^{st}\) step: \( y = \frac{3}{2t^2} + \frac{2}{t} = \frac{4t+3}{2t^2} ; \quad y_t' = \frac{dy}{dt} = \frac{4\cdot2t^2 - 4(4t+3)}{4t^4} = \frac{-2t-3}{t^3}; \)

\( x_t' = \frac{dx}{dt} = \frac{t^3 - 3t^2(1+t)}{t^6} = \frac{-2t-3}{t^4} \)

2\(^{nd}\) step: \( y_x' = \left(\frac{dy}{dx}\right) = \frac{dy_t'}{dx_t'} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{-2t-3}{t^3} \cdot \frac{-2t-2}{t^3} = \frac{-2t^3}{t^3} = \frac{-2t}{t^3} = \frac{-2t}{t^3} \cdot \frac{t^3}{t^3} = \frac{t^4}{t^3} = 1 + t \)

Final step: \( xy'^3 = \frac{1+t}{t^3} \cdot t^3 = 1 + t ; \quad 1 + y' = 1 + t \rightarrow xy'^3 = 1 + y'. \)

Q. E. D. (Qoud eram demonstrandum (Lat): Which was to be proved).

**Descartes' Rule of Signs** This rule helps to determine the number of positive, negative, and non real complex zeros of a polynomial function.

Let \( y = f(x) \) define a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of \( x \).

(a) The number of positive real zeros of \( y = f(x) \) either equals the number of variations in sign occurring in the coefficients of \( f(x) \), or is less than the number of variations by a positive even integer (2 or 4 or . . .).

(b) The number of negative real zeros of \( y = f(x) \) either equals the number of variations in sign occurring in the coefficients of \( f(-x) \), or is less than the number of variations by a positive even integer (2 or 4 or . . .).

**Example** Determine the different possibilities for the numbers of positive, negative, and non real complex zeros of the polynomial function

\( f(x) = 3x^4 - 5x^3 + 7x^2 - 4x - 2 \)

**Solution** We first consider the possible number of positive zeros (or roots) by observing that our polynomial function \( f(x) \) has three variations in signs.

\( f(x) = 3x^4 - 5x^3 + 7x^2 - 4x - 2 \) \quad (There are three variations in sign: 3→−5, −5→7, 7→−4)

Thus, according Descartes' rule of signs, \( f(x) \) has either three or one (since 3 − 2 = 1) positive real zeros.

For the number of negative zeros we will consider variations in signs of \( f(-x) \).

\( f(-x) = 3(-x)^4 - 5(-x)^3 + 7(-x)^2 - 4(-x) - 2 = 3x^4 + 5x^3 + 7x^2 + 4x - 2 \) \quad Since there is only one variation in sign, \( f(x) \) has exactly one negative real zero/root. Because \( f(x) \) is a polynomial function of 4\(^{th}\) degree, it must have four complex zeros/roots, some of which may be repeated. Descartes' rule of
signs showed that exactly one of these four complex zeros is a negative real number.

(a) One possible combination of the zeros/roots is one negative real zero/root, three positive real zeros/roots, and no non-real complex zeros/roots.

(b) Another possible combination of the zeros/roots is one negative real zero/root, one positive real zero/root, and two non-real complex zeros/roots.

**descending order** A polynomial in one variable written so that the exponents/powers of the variable decrease from left to right. For example, the polynomial $3x^4 - 4x^3 + 2x^2 + 7x - 5$ is a polynomial in descending order.

**descending sequence** A sequence where the numbers are decreasing from beginning to end.

**descriptive statistics** The branch of statistics that includes collecting and analyzing data to describe actual situation.

**determinant** The expression $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is termed a determinant of the second order (in contrast to determinants of orders three, four, etc.).

**deviation** The difference between a number in a set of data and the mean of all numbers in the set of data.

**diagonal of a polygon** A line segment that joins two nonadjacent vertices of a polygon.

**diagram** A drawing that represents a mathematical situation.

**diameter** A chord that passes through the center of a circle; twice the length of the radius of the circle.

**Didactics of Mathematics** Didactics of Mathematics is the science of how mathematics is shared and learned in various institutions. According to Juan Godino and Carmen Batanero (1997), this science can be defined also as the scientific and scholarly field of research which aims to identify, characterize, and understand the phenomena and processes conditioning the teaching and learning of mathematics.

**difference** The result obtained when numbers or expressions are subtracted. The difference of $a$ and $b$ is defined as $a - b$.

**difference of two cubes** $a^3 - b^3$. This algebraic expression can be expressed in the factored form as the special product, i.e., $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

**difference quotient/average rate of change** The difference quotient or the average rate of change of a function $f(x)$ with respect to $x$ for a function $f$ as $x$ changes from $x$ to $x + h$ is

$$\frac{f(x+h) - f(x)}{h}$$

*Bob Dylan:* “There is nothing so stable as change.”
Example Determine the difference quotient for \( f(x) = 6x^2 + 10x - 15 \).

1st step: Find \( f(x + h) \) (Hint: you will substitute \( x + h \) for \( x \))

\[
f(h + h) = 6(x + h)^2 + 10(x + h) - 15 = 6(x^2 + 2hx + h^2) + 10x + 10h - 15 = 6x^2 + 12hx + 6h^2 + 10x + 10h - 15
\]

2nd step: Find the difference \( f(x + h) - f(x) \)

\[
f(x + h) - f(x) = 12hx + 6h^2 + 10h \quad \text{(Factor out \( h \))}
\]

3rd step: \( \frac{f(x+h) - f(x)}{h} = \frac{h(12x+6h+10)}{h} = 12x + 6h + 10. \)

difference of two sets \( A \) and \( B \) The difference of two sets \( A \) and \( B \), symbolized \( A \setminus B \) or \( A - B \), is the set of elements of the set \( A \), which do not belong to the set \( B \), i.e. \( A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \} \).

difference of two squares \( a^2 - b^2 \) This expression can be expressed in the factored form as the special product, i.e. \( a^2 - b^2 = (a + b)(a - b) \).

differentiable function A function having derivative at some point is called differentiable at this point.

differential equation Any equation involving \( \frac{dy}{dx} \), or, equation in which the unknown function appears under sign of the derivative or the differential. The order of a differential equation is the highest order of the derivative or differential of the unknown function. Usually a solution of an equation is a number, but the solution of the differential equation is an function/equation.

Example Solve the differential equation: \( \frac{dy}{dx} = 3x^2 + 4. \)

Solution To solve any differential equation we will use integration.

\[
\frac{dy}{dx} = 3x^2 + 4 \quad \rightarrow \quad dy = (3x^2 + 4)dx \quad \int (\text{integration}) \quad \rightarrow \quad y = \int (3x^2 + 4)dx = 3 \int x^2dx + 4 \int dx = x^3 + 4x + C, \quad \text{where } C \text{ is an arbitrary constant.}
\]

Equation \( y = x^3 + 4x + C \) is called the general solution of the given differential equation.

differentiation The operation of finding the derivative of the given function.

digit A number 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 that fills or names a place-value location. The word “digit” for the basic numerals in 'our' (Hindu) number system is deriving from the Latin word digitus for finger.

dilatation A transformation in which every point \( P \) has an image point \( P' \) such that a line connecting the two points passes through a point \( O \), known as the center of dilatation, and \( OP' = k \cdot OP \), where \( k \) is the
scale factor of the dilatation. In other words, dilatation is proportional shrinking or enlargement of a figure.

dimension The dimension* of a space (or object) is defined as the minimum number of coordinates needed to specify each point within it. For example, a straight line has a dimension of one because only one coordinate is needed to specify a point on it.

","*“The notion of dimension belongs to the most fundamental mathematical ideas. In Western civilization and our school system, we become exposed in early life to assertion that the dimension of our physical space is three (and somewhat later, the time furnishes the fourth dimension).”


direction angle of a vector The positive angle between the x-axis and a position vector (a vector with initial point at the origin) is the direction angle for the given vector.

direction of vector The orientation of a vector, generally indicated by an arrowhead ( \vec{a} ).

direct variation A function whose equation has the form \( y = kx \), where nonzero number \( k \) is so called the constant of variation. N. B. In direct variation, an assignment of increasing absolute values for \( x \) produces increasing absolute value for \( y \).

directrix A fixed straight line that, together with a focus, is used to determine the points that forms a parabola.

discount An amount subtracted from the regular price of an item. In other words, the discount is the difference between the original price and the sale price.

discount rate The percent of decrease in the regular price of an item.

discrete mathematics Discrete mathematics is the branch of mathematics dealing with objects/sets that can assume only distinct separated values. The term “discrete mathematics” is therefore used in contrast with “continuous mathematics”, which is the branch of mathematics dealing with objects/sets that can vary smoothly. Whereas discrete objects/sets can often be characterized by whole numbers or integers, continuous objects/sets require real numbers.

discrete set A set which made up only of isolated points (See isolated points) is called a discrete set. For example, the set of natural numbers, i.e. \( N = \{1, 2, 3, \ldots \} \), is a discrete set.

disjoint sets Two sets that have no elements in common.

disjunction A sentence in which two or more sentences are joined by the word or to make a compound sentence.
**dispersion** Variation of data from the mean.

**distance formula** The distance, $d$, between any two points $(x_1, y_1)$ and $(x_2, y_2)$ is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**distributive property of multiplication over addition/subtraction** If $a$, $b$, and $c$ are real numbers, then $a(b + c) = ab + ac$ or $a(b - c) = ab - ac$.

**divergent sequence** If an infinite sequence does not converge to some number, then it is called a divergent sequence.

**dividend** The original number into which the divisor is divided.

**dividing a nonzero number by zero** The quotient obtained by dividing some nonzero number by zero does not exist because in this case no number can satisfy the definition of a quotient. As an example, we can take $\frac{9}{0}$ (or $9 \div 0$). No matter what number we take to test out (say, 4, 5, 6) we get the same unsatisfactory answer ($4 \cdot 0 = 0$, $5 \cdot 0 = 0$, $6 \cdot 0 = 0$) whereas what we need is 9. Shortly speaking, dividing a nonzero number by zero is undefined. We can say that the problem of dividing a nonzero number by zero has no solution.

**divisible** (See ‘Tests for divisibility’) The number $b$ is said to be divisible by another number $a$ if $b$ is a multiple of $a$.

**division** It is the process of determining how many times one number (divisor) is contained in another (dividend). The answer obtained by division is called the **quotient**. In other words, division is repeated subtraction or the reverse of multiplication.

**divisor** The number divided into the dividend. For example, $40 \div 5 = 8$, the number $40$ is **dividend**, 5 is **divisor**, and 8 is the **quotient**.

**dodecahedron** A dodecahedron is a polyhedron with 12 faces, 20 vertices, and 30 edges.

**domain** The set of all first coordinates (or x-values) of the ordered pairs in a function or the set of all inputs. In other words, what can go into a given function is called domain.

*How to Find the Domain of a Function*

The domain, as we saw above, is defined as the full set of x-values that can be plugged into a function to produce a y-value. What kind method we will use for finding a domain it depends of the type of the given function.

1. **A polynomial function**
   For the polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, there are no restrictions and the domain the set of all real numbers.

2. **A rational function**
To find domain for this type of function of the form \( f(x) = \frac{g(x)}{h(x)} \), set the denominator equal to zero and exclude the x value(s) you find when you solve the equation \( h(x) = 0 \) (because division by zero is undefined).

**Example** Find the domain of the given function \( f(x) = \frac{3x - 1}{x^2 - 9} \).

**Solution** \( x^2 - 9 = 0 \) \( \Rightarrow \) \((x + 3)(x - 3) = 0 \) \( \Rightarrow \) After using zero-factor property: \( x_1 = 3, \ x_2 = -3 \) Therefore for the given rational function \( x \neq -3 \), and \( x \neq 3 \).

Hence

Domain = \((- \infty, -3) \cup (-3, 3) \cup (3, \infty)\).

3. A irrational/radical function

For all irrational/radical function with even roots/radicals we have standard restriction: the radicand (expression under radical sign) must be nonnegative real number because even roots of a negative number don’t exist. When the index of the radical expression is odd (odd root), the domain is the set of all real numbers.

**Example** Determine the domain of the function given by \( f(x) = \sqrt[4]{3 - 6x} \).

**Solution** \( 3 - 6x \geq 0 \) (because we have here 4th root) \( \Rightarrow \) \( x \leq \frac{1}{2} \) \( \Rightarrow \) Domain = \((- \infty, \frac{1}{2}]\)

4. A logarithmic function

For all logarithmic functions we have standard restriction that the logarithmand (expression under logarithmic sign) must be positive real number.

**Example** Find the domain of the logarithmic function \( y = \log(x^2 + 3x + 2) \).

**Solution** The logarithmand must be positive real number, i.e. \( x^2 + 3x + 2 > 0 \) (The solution of this quadratic inequality will be domain of the given logarithmic function)

1st step: We have to solve corresponding quadratic equation \( x^2 + 3x + 2 = 0 \)\n
\( x^2 + 3x + 2 = 0 \) \( \Rightarrow \) \((x + 2)(x + 1) = 0 \) \( \Rightarrow \) \( x_1 = -2, \ x_2 = -1 \) \( \Rightarrow \) Domain = \((- \infty, -2) \cup (-1, \infty)\)

dot (scalar or inner) product of two vectors

(See Scalar product of two vectors) The dot product of the two vectors \( \mathbf{u} = \langle a, b \rangle \) and \( \mathbf{v} = \langle c, d \rangle \) is denoted \( \mathbf{u} \cdot \mathbf{v} \), read “\( \mathbf{u} \) dot \( \mathbf{v} \)”, and given by \( \mathbf{u} \cdot \mathbf{v} = ac + bd = |\mathbf{u}| |\mathbf{v}| \cos \theta \), where \( \theta \) is the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \).

N. B. The dot product of two vectors is a real number/scalar, not a vector.

double integral

See “Multiple integral”.

doubling time

The amount of time necessary for a population/quantity to double in size.

duration

Another term for “amount of time” or “time interval”.

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E

e An irrational number* \( (e \approx 2.718281828, \ see \ transcendental \ number) \).
The mathematical constant \( e \) is a relative newcomer on the mathematics scene, but at the same time, the unique positive real number such that

\[
\begin{align*}
1) \quad &\int_1^e \frac{dt}{t} = 1; \\
2) \quad &e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots; \\
3) \quad &e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.
\end{align*}
\]

This transcendental number occurs naturally in any situation where a quality increases at a rate proportional to its value (a bank account producing interest, or a population increasing as its members reproduce, or ...). The number \( e \) is sometimes called Euler’s number after the famous Swiss mathematician **Leonhard Euler (1707 – 1783)** introduced this constant to the mathematical world. It is obviously one of the most important numbers in mathematics and in our everyday life, alongside the additive and multiplicative identities 0 and 1, the constant \( \pi \), and imaginary unit \( i \). By the way, Leonhard Euler, in his so beautiful and elegant equation/identity \( e^{ix} + 1 = 0 \), was connecting in an astonishingly simple way, from then to eternity, these five (or six) fundamental and seemingly unrelated numbers with celebrity status in mathematics (and also the three most important operations – addition, multiplication, and exponentiation). Just the single most beautiful and quintessential equation in all of mathematics. For some nonmathematical and mathematical people this equation is like **Equation of God**. The readers of the *Mathematical Intelligencer* voted it “the Most Beautiful Mathematical Formula Ever”. Since \( e \) (like \( \pi \)) is transcendental, therefore irrational number, its value will be given exactly as a finite or eventually repeating decimal. In its infinite, aperiodic decimal expansion this number will be in form: \( e = 2.7182818284\ldots \) This non-algebraic number occurs naturally in any situation, especially as base of the natural logarithm, where a quantity increases at a rate proportional to its value (a bank account producing interest, or a population increases as its members reproduce,...). With this number we have also an mystery: Is \( e^x \) algebraic number or no? For \( t^i \) Euler’s amazing and divine identity (parent of \( e^{ix} + 1 = 0 \)) from 1748:

\[
e^{ix} = \cos x + i\sin x,
\]

give us answer.

Namely, \( i^i = e^{- \frac{\pi}{2}} \approx 0.207879576350761908546955 \).
eccentricity  The eccentricity of a parabola, ellipse, or hyperbola is the fixed ratio. For parabola it is ratio of the distance from a point P to a focus compared to the distance from the same point to the directrix. The eccentricity of the ellipse and hyperbola is the constant ratio \( \frac{c}{a} \), where \( a \) is the distance from the center to a vertex, and \( c \) is the distance from the center of the figure to a focus.

economic lot size  The number or quantity of material or units of a manufactured good that can be produced or purchased within the lowest unit cost range over given period, usually annual. It is determined by reconciling the decreasing unit cost of handling, storage, insurance, interest, etc. Also known as project life. Calculus can help to find this number \( q^* \) of units that should be manufactured in each production lot or batch in order to minimize the total annual manufacturing cost \( T(q) \).

\[
\begin{align*}
T(q) &= \frac{fM}{q} + gM + \frac{kq}{2}, \\
T'(q) &= -\frac{fM}{q^2} + \frac{k}{2} = 0 \Rightarrow q = \sqrt{\frac{2fM}{k}} \quad (\text{The economic lot size that minimizes total production costs})
\end{align*}
\]

edge  The line segment where two faces of a solid figure meet.

Elasticity of Demand  Let \( q = f(p) \), where \( q \) is demand at a price \( p \). The elasticity of demand is given by equation/formula \( E = -\frac{p}{q} \cdot \frac{dq}{dp} \).

1. Demand is inelastic if \( E < 1 \) (the relative change in demand is less than the relative change in price, and total revenue increases as price increases).
2. Demand is elastic if \( E > 1 \) (the relative change in demand is greater than relative change in price, and total revenue decreases as price increases).
3. Demand has unit elasticity if \( E = 1 \) (the percentage changes in price and demand are relatively equal, and total revenue is maximized).

element/member  Each object belonging to a set.

elements of a matrix  The individual numbers in a matrix; also called entries. For example in a two-by-two matrix \[
\begin{bmatrix}
2 & 7 \\
11 & 14
\end{bmatrix}
\] numbers 2, 7, 11, and 14 are elements (or entries) of the given matrix.

Elimination Method  A method of solving systems of equations using addition/subtraction principle to solve a system of equations. As Leonhard Euler remarked (1771), this method (also known as “Gaussian elimination”) is the most natural way of proceeding. When we are using this method, the central idea is to eliminate one of the variables. To do that, one of the variables in two equations must have opposite coefficients.
Example 1  Solve the system using the elimination method.

\[ 5x - y = 5, \quad \ldots\ldots \quad (1) \]
\[ 3x + y = 11 \quad \ldots\ldots \quad (2) \]

**Solution**  The key to advantage of the elimination method for solving this system involves the \( y \) in the first equation and \( -y \) in the second. The terms opposites, and after adding equations we will eliminate this variable, because the sum of two opposites is 0. Therefore,

\[ 5x - y = 5, \]
\[ 3x + y = 11 \quad \text{(Adding)} \]
\[ 8x = 16, \quad \text{We have “eliminated” one variable. This is why we call this the elimination method.} \]
\[ x = \frac{16}{8} = 2 \]

Next step, we substitute 2 for \( x \) in either of the original equations.
\[ 3x + y = 11, \quad \text{Equation} \quad (2) \]
\[ 3(2) + y = 11 \]
\[ 6 + y = 11 \quad \text{(Isolate} \ y \text{by transposing 6 on other side)} \]
\[ y = 11 - 6 \]
\[ y = 5. \]

We will check the ordered pair \((2, 5)\).
\[ 5x - y = 5 \]
\[ 5(2) - 5 = 10 \]
\[ 3(2) + 5 = 11 \]
\[ 10 - 5 = 5 \quad \text{(True)} \]
\[ 5 = 5 \quad \text{(True)} \]

Since \((2, 5)\) checks, it is the solution.

**Example 2**  Solve the system using the elimination method.

\[ 2x + 3y = -1 \quad \text{(1)} \]
\[ 3x + 5y = -2 \quad \text{(2)} \]

**Solution**  We have to decide which variable to eliminate. Coefficients of the \( x \)-terms are smaller numbers (2 and 3), and we will decide to eliminate the \( x \)-term. The least common multiple of 2 and 3 is 6. That means we will multiply the equation (1) by \(-3\) and the equation (2) by \(2\). Then we will add equations and we will eliminate the \( x \)-variable and we will have only one equation that can be solved for \( y \).

\[ -6x - 9y = 3 \]
\[ 6x + 10y = -4 \]
\[ y = -1 \]

We substitute \( -1 \) for \( y \) in the equation (1).
\[ 2x + 3(-1) = -1 \]
\[ 2x - 3 = -1 \quad \text{(Isolate the} \ x \text{-term by transposing} -3 \text{on other side of the equation)} \]
\[ 2x = -1 + 3 \]
\[ 2x = 2 \quad \text{(isolate} \ x \text{by transposing 2 on other side, using division:} \ x = \frac{2}{2} ) \]
\[ x = 1 \]

We will check ordered pair \((1, -1)\).
\[ (1): \quad 2(1) + 3(-1) = 2 - 3 = -1 \quad \text{(True)} \]
\[ (2): \quad 3(1) + 5(-1) = 3 - 5 = -2 \quad \text{(True)} \]

Since \((1, -1)\) checks, it is the solution.

**Example 3**  Solve using the elimination method.
Solution We will select any two of the three equations and work to get an equation in two variables. Let's add equation (1) and (3):

\[
x + 3y - 3z = 12 \\
-x + 2y - z = 1
\]

\[
5y - 4z = 13
\]  \( \text{(4)} \) (Adding to eliminate \( x \))

In the next step we will select a different pair of equations and eliminate the same variable that we did in the first step.

We will multiply equation (3) by 3:

\[
- x + 2y - z = 1 / \cdot 3
\]

\[
-3x + 6y - 3z = 3
\]  \( \text{(5)} \)

\[
3x - y + 4z = 0
\]  \( \text{(2)} \) (By adding equations (5) and (2) we will eliminate \( x \))

\[
5y + z = 3
\]  \( \text{(6)} \)

Now we get the system of two equations in two variables.

\[
5y - 4z = 13 / \cdot (-1)
\]  \( \text{(4)} \) [Multiply equation (4) by (-1)]

\[
5y + z = 3
\]  \( \text{(6)} \)

\[
-5y + 4z = -13
\]

\[
5y + z = 3
\]  \( \text{(6)} \)

\[
5z = -10 \Rightarrow z = -2
\]

We can use either equation (4) or (6) to find \( y \). We choose (6).

\[
5y + z = 3 \Rightarrow 5y - 2 = 3 \Rightarrow 5y = 5 \Rightarrow y = 1
\]

We now have \( y = 1 \) and \( z = -2 \). To find value for \( x \), we can use any of the original three equations and substitute our solutions for \( y \) and \( z \) to find the value for \( x \). Let's use the equation (1) and substitute our two numbers in it:

\[
x + 3y - 3z = 12
\]  \( \text{(1)} \)

\[
x + 3 + 6 = 12 \Rightarrow x + 9 = 12 \Rightarrow x = 12 - 9 \Rightarrow x = 3
\]

The solution is the ordered triple \((3, 1, -2)\).

**ellipse** (See 'conic sections') The set of all points in the plane for each of which the sum of the distances from two fixed points in the same plane is a constant. Each fixed point is called a **focus** (plural, **foci**) of the ellipse, and the distance between them is the **focal length**.

**ellipsis** An ellipsis is the series of three dots that we use to show that a pattern of number continues. For example, the number 3.1313 . . . includes an ellipsis.
empirical probability  Determining the chance of something happening in the future by observing past results is called empirical probability. The empirical probability of an event $E$, $P(E)$, can be determined by the following formula.

$$P(E) = \frac{\text{number of times even } E \text{ has occurred}}{\text{total number of times the experiment has been performed}}$$

empty set (null set) The empty set (or null set), written $\emptyset$ or $\{\}$, is the set containing no elements.

N. B. The set $\{\emptyset\}$ is not the empty set.

end behavior  The end behavior of a graph of a polynomial function describes how the values of $y$ increase or decrease as $|x|$ increases without bound.

enumeration  An enumeration of a collection/set of items is a complete, ordered listing of all of the items in that collection/set.

equality  The state of being equal; shown by the equal sign.

equally likely outcomes  If each outcome of an experiment has the same chance of occurring as any other outcome, we say that the outcomes are equally likely outcomes.

equation  A statement that two expressions are equal. For example, $3x + 4 = x - 2$.

equations of the straight line:
1) Standard Form (or Implicit Form) $Ax + By = C$;
2) Slope-intercept Form (or Explicit Form) $y = mx + b$;
3) Point-slope Form $y - y_1 = m(x - x_1)$;
4) Intercept-Intercept Form $\frac{x}{a} + \frac{y}{b} = 1$, where $a$ is x-intercept, and $b$ is y-intercept;
5) Point-Point Form - Equation of a Straight Line Passing Through Two Given Points $(x_1, y_1)$ and $(x_2, y_2)$: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$;
6) Horizontal line: $y = k$;
7) Vertical line: $x = k$

equiangular polygon  A polygon in which all angles are congruent.

equidistant  Equally distant. For example, all points on a circle are equidistant from its center.

equilateral/regular polygon  A polygon in which all sides are congruent.

equilibrant  The opposite vector of the resultant of two vectors is called the equilibrant.

equilibrium point  The point of intersection between the demand function $D(q)$ and the supply function $S(q)$. To find the equilibrium point we will solve equation $D(q) = S(q)$. 

equilibrium price of the commodity  The price $p$ at the point where the supply $S(q)$ and demand $D(q)$ graphs for that commodity intersect, i.e. $D(q) = S(q)$. In other words, this is the price when the supply of goods in a particular market matches demand.

equilibrium quantity The equilibrium quantity is found when the prices for both supply and demand are equal, i.e. $S(q) = D(q)$, where $q$ is produced quantity.

**Example**  Let the supply and demand functions for butter pecan ice cream be given by $p = S(q) = \frac{2}{5}q$ and $p = D(q) = 100 - \frac{2}{5}q$, where $p$ is the price in dollars and $q$ is the number of 10-gallon tubs.

Find the equilibrium quantity and the equilibrium price.

**Solution** If we have the equilibrium quantity, then $S(q) = D(q) \rightarrow \frac{2}{5}q = 100 - \frac{2}{5}q \rightarrow (after grouping terms with q) \rightarrow \frac{4}{5}q = 100 \quad (after clearing the fraction \frac{4}{5} by multiplying both sides by its reciprocal, i.e. \frac{5}{4}) \rightarrow q = 125 \rightarrow S(125) = D(125) = 50$.

The equilibrium quantity is 125 10-gallon tubs of butter ocean ice cream, the equilibrium price is $50.

equilibrium values/rest points If $\frac{dy}{dx} = g(y)$ is an autonomous differential equation, then the values of $y$ for which $\frac{dy}{dx} = 0$ are called equilibrium values or rest points.

**Example** Identify the equilibrium values of $\frac{dy}{dx} = y^3 - y$.

**Solution** $\frac{dy}{dx} = 0 \Rightarrow y^3 - y = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y(y + 1)(y - 1) = 0 \Rightarrow y_1 = 0, y_2 = 1, y_3 = -1$.

Therefore, the equilibrium values are $y_1 = 0$, $y_2 = 1$, and $y_3 = -1$.

equivalence relation  $\sim$ Any binary relation $\sim$ in a set $A$ that satisfies the reflexive, symmetric, and transitive properties. Equivalently, for all $a, b$ and $c$ in $A$:

1) $a \sim a$ (Reflexivity);
2) If $a \sim b$ then $b \sim a$ (Symmetry);
3) If $a \sim b$ and $b \sim c$ then $a \sim c$ (Transitivity).

equivalent equations Two or more equations with the same solutions.

equivalent expressions The expressions that have the same value for all meaningful replacements.

equivalent inequalities Two or more inequalities that have the same solution set.

equivalent sets Two sets $A$ and $B$ are said to be equivalent if they contain the same number of elements, i.e. $n(A) = n(B)$.

equivalent systems of equations Systems of equations with the same solutions.

estimate To find an approximate answer.
estimation The estimation is the process of arriving at an approximate answer to a question. In other words, this is a way of approximating the answer to a question quickly.

eternity* Existence without beginning or end. Algebraically speaking, eternity as timelessness is a function of time.

* John Milton (1608-1674): “Eternity whose end no eye can reach.”

Euler’s Formula /Identity The formula $e^{ix} = \cos x + i \sin x$ is called Euler’s formula. This famous formula/identity establishes the profound relationship between trigonometric functions and the complex exponential/logarithmic functions, where $e$ is the base of the natural logarithm, $i$ is the imaginary unit, and $\cos$ and $\sin$ are the trigonometric functions cosine and sine respectively with the angle/argument $x$ is any real number, given in radians. The formula is still valid if $x$ is a complex number.

Example Compute $e^{2\pi i}$, $e^{-\pi i}$, $e^{\frac{\pi i}{2}}$, and $e^{\frac{3\pi i}{2}}$.

Solution By Euler’s formula, we will have:

- $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i \cdot 0 = 1;$
- $e^{-\pi i} = \cos (-\pi) + i \sin (-\pi) = \cos \pi - i \sin \pi = -1 - i \cdot 0 = -1;$
- $e^{\frac{\pi i}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i \cdot 1 = i;$
- $e^{\frac{3\pi i}{2}} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = 0 + i (-1) = -i.$

Euler’s constant $\gamma$ The Euler’s constant (also called Euler-Mascheroni constant) is a numerical constant recurring in mathematical analysis/calculus and number theory, usually denoted by the lowercase Greek letter gamma $\gamma$. This constant is defined as the limit of the difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) \approx 0.5772156649 \ldots$$

Euler’s number $e$ (See $e$)

Evaluating an Expression You evaluate an expression by replacing the variable(s) with the given number(s) and performing the indicated operations (using order of operations).

even function The function $y = f(x)$ is even if $f(-x) = f(x)$, $\forall x \in D$, where $D$ is domain of $f(x)$. In addition, its graph of even function is symmetric with respect to the y-axis.

even number A number divisible by 2 is called an even number, otherwise it is odd.

even root A root with an even index. For example, $\sqrt[4]{a}$, and $\sqrt[6]{a}$ are even roots.

event An event is a subcollection/subset of the results or outcomes of an experiment.
**evolution** Finding the root of a number. Finding (or extracting the root of a number is a process by which we find the base of a power from the power and exponent. *Involution and evolution* (raising a number to an integral power), like *addition* and *subtraction*, or *multiplication* and *division*, are pairs of inverse operations.

**exclusive OR** Indicating either one or the other, but not both.

**expanded notation/form** A way to write numbers that shows the place value of each digit. For example, expanded notation for the number 278:

\[278 = 2 \text{ hundreds} + 7 \text{ tens} + 8 \text{ ones}\]

**expansion** A dilatation in which the preimage is enlarged in size.

**expected value** Suppose the random variable \( x \) can take on the \( n \) values \( x_1, x_2, \ldots, x_n \). Also, suppose the probabilities that these values occur are respectively \( p_1, p_2, \ldots, p_n \). Then the expected value of the random variable is

\[
E(x) = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_n p_n,
\]

where symbol \( p_1 \) represents the probability that the first event will occur, and \( x_1 \) actually represents the net amount gain or lost if the first event occurs, \( p_2 \) is the probability of the second event, and \( x_2 \) is the net amount gained or lost if the second event occurs, and so on.

**experiment** An experiment is controlled operation that yields a set of results.

**exponent** *(See SI Prefixes)* A number that indicates how many times a base is repeated. In the exponential expression or power \( a^n \), \( n \) is the exponent, and \( a \) is the base. In other words, exponents are another way (short cut) to write multiplication, i.e. \( a^n = a \cdot a \cdot a \cdot \cdots \cdot a \) (note there are \( n \) \( a \)'s in the product).

**exponential equation** An equation in which a variable (unknown quantity) appears as an exponent or part of an exponent.

**Example 1** Solve \( 2^{x-5} = 32 \).

**Solution** This type of the exponential equation we can solve using a common base on both sides of the equation.

\[2^{x-5} = 32 \quad \text{(Write 32 as a power of 2)}\]

\[2^{x-5} = 2^5 \quad (a^m = a^n \text{ if and only if } m = n, \text{ for any real number } a > 0, a \neq 1) \Rightarrow x - 5 = 5 \Rightarrow x = 5 + 5 \Rightarrow x = 10.\]

The solution set is \( \{10\} \).

**Example 2** Solve \( 3^x = 25 \).

**Solution** For this type of the exponential equation we can use only properties of logarithms. By using common logarithms, we can proceed as follows:

\[3^x = 25/\log\]

\[\log 3^x = \log 2^5 \quad (\log a^n = n \log a)\]

\[x \log 3 = 2 \log 5 \quad (ax = b \Rightarrow x = \frac{b}{a})\]
The solution set (to the nearest hundredth) is {2.93}.

Example 3 Solve \(4^x + 6^x = 9^x\).

Solution This type of the exponential equation is the special ‘mixture’ of the equations in the previous two examples. \(4^x + 6^x = 9^x/6^x\)

By dividing equation by \(6^x\) we will have:

\[
\frac{4^x}{6^x} + \frac{6^x}{6^x} = \frac{9^x}{6^x} \quad \text{or} \quad \left(\frac{4}{6}\right)^x + 1 = \left(\frac{9}{6}\right)^x \quad \text{or (after simplifying)} \quad \left(\frac{2}{3}\right)^x + 1 = \left(\frac{3}{2}\right)^x
\]

Let \(\left(\frac{2}{3}\right)^x = u \quad \ldots (1) \Rightarrow \left(\frac{2}{3}\right)^x = u^{-1} \quad \text{(Substitution method)} \Rightarrow u + 1 = u^{-1} \quad \text{or}
\]

\[u + 1 = \frac{1}{u} \quad \text{(After clearing fraction)} \Rightarrow \]

\[u^2 + u = 1 \quad \text{(We will write this quadratic equation in the standard form and solve using quadratic formula)} \quad u^2 + u - 1 = 0, \quad a = 1, b = 1, c = -1
\]

\[u_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow u_1 = \frac{-1 + \sqrt{5}}{2}; \quad u_2 = \frac{-1 - \sqrt{5}}{2}.
\]

We can see that \(u_1 > 0\), and \(u_2 < 0\).

Replace \(u_1\) with \(\frac{-1 + \sqrt{5}}{2}\) in equation (1).

\[
\left(\frac{2}{3}\right)^x = \frac{-1 + \sqrt{5}}{2} \quad \text{in} \quad \text{(See Example 2)} \Rightarrow
\]

\[x \log\left(\frac{2}{3}\right) = \log \left(\frac{-1 + \sqrt{5}}{2}\right) \quad \text{(Quotient Property of Logarithms:} \quad \log \frac{a}{b} = \log a - \log b) \quad \Rightarrow
\]

\[x(\log 2 - \log 3) = \log (-1 + \sqrt{5}) - \log 2 \quad \text{or}
\]

\[x = \frac{\log(-1 + \sqrt{5}) - \log 2}{\log 2 - \log 3}
\]

We will discard \(u_2\) because \(\left(\frac{2}{3}\right)^x\) is always positive real number. In other words, the equation \(\left(\frac{2}{3}\right)^x = \frac{-1 - \sqrt{5}}{2}\) has no solution.

exponential growth/decay An increase/decrease in quantity over time that can be modeled by an exponential function of the form \(A(t) = A_0 e^{kt}\), where \(k\) is a constant and \(A_0\) is initial amount, or number of some quantity present at time \(t = 0\). In other words, this function models a situation in which a quantity changes at rate proportional to the amount present at any time \(t\).

exponential notation of an n-th root \(\sqrt[n]{x^m} = x^{m/n}\).

exponentiation An act/operation of raising a quantity to a power.

expression A sentence that usually includes operations, variables, and numbers.
**extraneous solution** When an equation is transformed, for example by squaring both sides, the transformed equation may have solutions that are not solutions to the original equation. These solutions are called extraneous.

**extrapolation** The process of predicting a future value on the basis of given data.

**Extrema of a Function** The function \( y = f(x) \) is said to have a relative/local maximum (minimum) at the point \( x_0 \) if there is a neighborhood of the point \( x_0 \) in which inequality \( f(x) < f(x_0) \) \( [f(x) > f(x_0)] \) respectively for \( x \neq x_0 \).

**Extreme Value Theorem** A function \( f(x) \) that is continuous on a closed interval \( [a, b] \), has both an absolute maximum* and an absolute minimum on this interval.

*Let \( f(x) \) be a function defined on some interval \( [a, b] \). Let \( x_0 \in [a, b] \). Then \( f(x_0) \) is the absolute maximum of \( f(x) \) on the interval \( [a, b] \) if \( f(x) \leq f(x_0) \), \( \forall x \in [a, b] \), and \( f(x_0) \) is the absolute minimum of \( f(x) \) on the interval \( [a, b] \) if \( f(x) \geq f(x_0) \), \( \forall x \in [a, b] \).

**extremes** The first and lest terms in the proportion. In the proportion \( \frac{a}{b} = \frac{c}{d} \), for example, the extremes are \( a \) and \( d \) (\( b \) and \( c \) are the means).

**Factored completely** A polynomial is factored completely when it is written as product of prime polynomials.

**factors** The numbers and variables/expressions that are multiplied in an expression. To factor a polynomial is to express it as a product. For example, \( 7, 2x + 5 \) and \( x - 3 \) are factors of the polynomial \( 7(2x + 5)(x - 3) \).

**factoring** Writing an expression/number as a product.

**factoring quadratic trinomials of type \( ax^2 + bx + c \) by using ac-method**

1. Make sure that any common factors have been factored out.
2. Multiply the leading coefficient \( a \) and the constant \( c \).
3. Find a pair of factors of \( ac \), \( p \) and \( q \), such that \( p + q = b \) and \( pq = ac \).
4. Rewrite the middle term of the trinomial, \( bx \), as \( px + qx \).
5. Factor by grouping.
Example  Solve by factoring:  \(3x^2 - 16x - 12 = 0\).

Solution

Step 1-2) Factor given trinomial using \textbf{ac-method}: \(3x^2 - 16x - 12 = 0\); \(\rightarrow\) \(ac = -36\)

Step 3) \(ac = -36, \rightarrow p = -18, q = 2\);

Step 4) \(3x^2 - 18x + 2x - 12 = 0\);

Step 5) \(3(x - 6) + 2(x - 6) = 0\), Factor out common factor \(x - 6\):

\( (x - 6)(3x + 2) = 0\);

Final step: After getting factored form of our equation we will use \textbf{zero-factor property}:

\( x - 6 = 0 \) or \(3x + 2 = 0 \rightarrow x_1 = 6 \) or \(3x = -2\) and \(x_2 = \frac{-2}{3}\).

\textbf{factoring quadratic trinomial of type} \(x^2 + (m + n)x + mn\) For this case of the quadratic trinomial when \(a = 1, b = m + n\), and \(c = mn\) we will have \(x^2 + bx + c = x^2 + (m + n)x + mn = x^2 + mx + nx + mn = (\text{two groups}) = x(x + m) + n(x + m) = \) (factor out the common factor \(x + m\)) = \((x + m)(x + n)\).

Therefore, \(x^2 + (m + n)x + mn = (x + m)(x + n)\).

Example  Factor: \(x^2 + 5x + 6\).

\(x^2 + 5x + 6 = (x + 2)(x + 3)\) because \(5 = 2 + 3\) and \(6 = 2 \cdot 3\).

\textbf{factorial notation} This notation is a compact way of writing a product of the first \(n\) consecutive natural numbers. The symbol \(n!\) (read “\(n\)-factorial”) is defined as follows: \(n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n\).

Example  Solve: \(\frac{m! - (m - 1)!}{(m + 1)!} = \frac{1}{6}\).

Solution \(\frac{m! - (m - 1)!}{(m + 1)!} = \frac{1}{6} \Rightarrow \frac{\left(1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot m - 1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot 1 \cdot m \cdot (m + 1)\right)}{1 \cdot 2 \cdot 3 \cdots (m - 1) \cdot m \cdot (m + 1)} = \frac{1}{6} \Rightarrow \) (After factoring out the greatest common factor) \(\Rightarrow \frac{1}{\left[1 \cdot 2 \cdot 3 \cdots (m - 1)\right] \cdot (m - 1) \cdot 1 \cdot m \cdot (m + 1)} = \frac{1}{6} \Rightarrow \) (After canceling out the greatest common factors) \(\Rightarrow \frac{m - 1}{m(m + 1)} = \frac{1}{6} \Rightarrow\) (After using proportion property) \(\Rightarrow 6(m - 1) = m(m + 1)\)
or \(6m - 6 = m^2 + m\) or \(m^2 - 5m + 6 = 0\)

To solve this quadratic equation we will use quadratic formula \(m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; a = 1, b = -5, c = 6\).

\(m_{1,2} = \frac{5 \pm \sqrt{25 - 24}}{2} \Rightarrow m_{1,2} = \frac{5 \pm 1}{2} \Rightarrow m_1 = 3, \ m_2 = 2\)

\textbf{Factor Theorem}  The binomial \(x - k\) is a factor of the polynomial function \(f(x)\) if and only if \(f(k) = 0\). It means if \(k\) is zero of \(f(x)\) than \(f(k) = 0\).

\textbf{factor tree}  A way of showing how a number is factored into prime factors.

\textbf{Fair Price}  To determine the fair price, we will use the following formula.

\(\text{Fair price} = \text{expected value} + \text{cost to play}\)

\textbf{Fermat's* Last (Great) Theorem}  This Fermat’s theorem or conjecture is the world’s most famous mathematical problem. It states that equation \(x^n + y^n = z^n\) has no non-zero integer solutions for \(x, y, \) and \(z\) when \(n \geq 3\). In other words, \(x^n + y^n \neq z^n\), if \(n \geq 3\).
In 1637 famous French mathematician Pierre de Fermat (1601-1665) jotted this conjecture in the margins of an ancient Greek mathematical text with the remark that the proof is so simple, and vexed mathematicians whole 358 years before it was finally solved in by British mathematician Andrew John Wiles, with assistance of Richard Taylor. Wiles published his proof in 1995. By the way, Pierre de Fermat was French lawyer and an amateur mathematician, but for sure one of the greatest mathematicians of the seventeenth century. He formulated this theorem in 1637.

**Pierre de Fermat:** “It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general any power higher than the second power into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.”

**Fermat’s Little Theorem** In the expression \( x = \frac{2^{p-1} - 1}{p} \) we can see that \( x \) is always an integer if \( p \) is odd prime. This Fermat’s theorem gave us so interesting question: *For what value of \( p \) \( x \) is a perfect square?* For \( p = 7 \) we will get the first perfect square. Namely,

\[
x(7) = \frac{2^6 - 1}{7} = \frac{64 - 1}{7} = \frac{63}{7} = 9
\]

**Fermat number** Fermat conjectured that each number of the form \( 2^{2^n} + 1 \), now referred to as a *Fermat number*, was prime for each natural number \( n \). In 1732, Leonhard Euler proved that for \( n = 5 \), \( 2^{32} + 1 \) was a composite number, thus disproving this Fermat’s conjecture.

**Fermat’s principle in optics** This principle states that light* travels from one point to another along a path for which the time of travel is a minimum.

*The speed of light depends on the medium through which it travels, and is generally slower in denser media.

**Fermat’s theorem for extreme values of a function** If \( x_0 \) is a point of extremum of a function \( f(x) \), and the derivative \( f'(x_0) \) exists at this point, then it is equal to zero:

\[ f'(x_0) = 0. \]

**Fibonacci* sequence** A special sequence of numbers in which each number is the sum of the two preceding numbers: \( \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \} \).

*Leonardo Pisano Fibonacci (c.1170-c.1250), famous Italian mathematician, found this pattern.

**field** Any number system (set of numbers) with two operations (addition and multiplication) defined in which the field axioms hold. Or briefly, a field is a commutative ring. The most commonly used fields are the field of real numbers, the field of complex numbers, and the field of rational numbers. The theory of fields plays a essential and profound role especially in number theory and algebraic geometry.

**field axioms** The closure, commutative, associative, identity, inverse, and distributive properties.
Finding the Root of a Number Finding (or extracting) the root of a number is a process (operation) by which we find the base of a power from the power and the exponent. The power here is termed the radicand, the exponent is here the index* of the root, and desired base of the power is termed the root.

*The second root is usually called the square root, the third root the cube root. When taking square roots the index 2 is usually omitted: $\sqrt{16} = 4$ means $\sqrt[2]{16} = 4$.

finite set A finite set is a set that has limited number of elements (members).

First Derivative Test (the first sufficient condition for an extremum) Let $x_0$ be a critical number for a function $y = f(x)$. Suppose that $f(x)$ is continuous and differentiable on $(a, b)$ except possibly at $x_0$, and that $x_0$ is the only critical number for $f(x)$ in $(a, b)$. Then, if the derivative $f'(x)$ changes sign from plus to minus (from minus to plus) when passing through the point $x_0$ (in the direction of the increase of $x$), then function $y = f(x)$ has a relative/local maximum (minimum) at the point $x_0$. If the derivative of the function does not change sign when passing through the point $x_0$, then the function $y = f(x)$ does not posses an extremum at the point $x_0$.

Example 1 Find the $x$-value of all points where function defined as follows have any relative (local) extrema. Find the value(s) of any relative (local) extrema.

$$f(x) = \frac{x^2 - 6x + 9}{x + 2}$$

Solution We compute the derivative (quotient rule)

$$f'(x) = \frac{(2x-6)(x+2)-(1)(x^2 - 6x + 9)}{(x+2)^2} = \frac{x^2 + 4x - 21}{(x+2)^2} = \frac{(x+7)(x-3)}{(x+2)^2}$$

and find the critical numbers (stationary points): $f'(x) = 0 \rightarrow (x + 7)(x - 3) = 0$ (using zero-factor property) $\rightarrow x + 7 = 0$ or $x - 3 = 0 \rightarrow x_1 = -7$ or $x_2 = 3$.

Note that $f(x)$ and $f'(x)$ do not exist at $x = -2$ (for this number the denominator is 0), so the only critical numbers (stationary points) are $-7$ and $3$ and we will examine the behavior of $f'(x)$ on four intervals: $(-\infty, -7)$, $(-7, -2)$, $(-2, 3)$, and $(3, \infty)$.

It is easy to see that $f'(x) > 0$ on the interval $(-\infty, -7)$, because, for example, $f'(-8) = \frac{11}{36} > 0$, $f'(x) < 0$ on the interval $(-7, -2)$, and $(-2, 3)$, because $f'(-3) = -24 < 0$, and $f'(0) = -\frac{21}{4} < 0$, and $f'(x) > 0$ on interval $(3, \infty)$, because $f'(4) = \frac{11}{36} > 0$.

Just now, we can see that:

1) $f(x)$ is increasing on $(-\infty, -7)$ and $(3, \infty)$;
2) $f(x)$ is decreasing on $(-7, -2)$ and $(-2, 3)$.

Consequently, at the point $x_1 = -7$ the function $f(x)$ has a relative (local) maximum $f_{\text{max}}(-7) = -20$, and at the point $x_2 = 3$ a relative (local) minimum $f_{\text{min}}(3) = 0$.

Example 2 Assume that the total revenue received from the sale of $x$ items is given by

$R(x) = 30 \ln(2x + 1)$, while the total cost to produce $x$ items is $C(x) = \frac{x^2}{2}$.

Solution Find the number of items that should be manufactured so that profit, $P(x) = R(x) - C(x)$, is a
The *profit function* will be a maximum when the derivative of the profit function is equal to 0.

\[ P(x) = R(x) - C(x) = 30\ln(2x - 1) - \frac{x}{2} ; \]

\[ P'(x) = \frac{30}{2x+1} \cdot 2 - \frac{1}{2}, \quad \text{N. B.} \quad (\ln u)' = \frac{1}{u} \cdot u' \]

\[ P'(x) = \frac{60}{2x+1} \cdot 2 - 1 ; \quad P'(x) = 0 \rightarrow \frac{60}{2x+1} = \frac{1}{2} \rightarrow 2x + 1 = 120 \quad \text{(Proportion property: If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc) \]

So, \( x = \frac{119}{2} = 59.5 \).

Thus, the maximum profit occurs when \( x = 59.5 \) or, in a practical sense, when 59 or 60 items are manufactured. Both 59 and 60 give the same profit.

**First-Order Linear Differential Equation** A differential equation that can be written in the form

\[ \frac{dy}{dx} + P(x)y = Q(x) \quad \text{or} \quad y' + P(x)y = Q(x) \]

where \( P \) and \( Q \) are continuous functions of \( x \). In other words, a differential equation is called linear if it is of the first degree with respect to the required function \( y \) and all of its derivatives.

The general solution is given by

\[ y = \frac{\int v(x)Q(x)dx}{v(x)}, \quad \text{or} \quad y = \frac{1}{v(x)} \int v(x)Q(x)dx, \quad \text{where} \quad v(x) = e^{\int P(x)dx}, \text{called the integrating factor, and } v(x) > 0. \] If an initial condition is given, we will use it to find the constant \( C \) and the particular solution.

**Example 1** Solve the differential equation \( y' + 2xy = x \).

**Solution**

*Step 1:* The equation is already expressed in the standard form with \( P(x) = 2x \) and \( Q(x) = x \);

*Step 2:* The integrating factor is \( v(x) = e^{\int P(x)dx} = e^{\int 2x} = e^{x^2} ; \)

*Step 3:* To find the general solution we have to evaluate firstly the indefinite integral

\[ \int v(x)Q(x)dx = \int e^{x^2} xdx = \frac{1}{2} e^{x^2}; \]

*Step 4:* The general solution is given by \( y = \frac{\int v(x)Q(x)dx+C}{v(x)} = \frac{e^{x^2}+C}{e^{x^2}} = \frac{1}{2} + \frac{c}{e^{x^2}} \).

**N. B.** The general solution gives all solutions of the equation (there are infinitely many, one for each value of \( C \)).

**Example 2** Find the particular solution of the differential equation \( y' + (\tan x)y = \cos^2 x, \quad y(0) = 1 \).

**Solution**

*Step 1:* We can see that the given equation is already in the standard form, and \( P(x) = \tan x \), and \( Q(x) = \cos^2 x \);

*Step 2:* We will find the integrating factor \( v(x) = e^{\int \tan x} = e^{-\ln \cos x} = e^{\ln \sec x} = \sec x ; \)
\( N.B. \quad e^{\ln a} = a \)

Step 3: We have to evaluate the indefinite integral

\[
\int v(x)Q(x)dx = \int \sec x \cos^2 x dx = \int \cos x dx = \sin x;
\]

Step 4: The general solution is given by

\[
y = \frac{\int v(x)Q(x)dx + C}{v(x)} = \frac{\sin x + C}{\sec x} = (\sin x + C)\cos x;
\]

Step 5: Using the initial condition we will find value of the constant C and the particular solution.

\( y(0) = 1 \Rightarrow 1 = (\sin 0 + 1)\cos 0 \Rightarrow C = 1. \)

Therefore, the particular solution is \( y = (\sin x + 1)\cos x \).

**fixed cost** In business, cost that must be paid whether or not a product is produced. In other words, the fixed cost is the cost if zero units are made. That is the cost, for example, for designing the product, setting up a factory, training workers, and so on.

**floor and ceiling functions** The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively. More precisely, floor function, denoted by \( f(x) = \lfloor x \rfloor \), is the largest integer not greater than \( x \) and ceiling function \( f(x) = \lceil x \rceil \) is the smallest integer not less than \( x \).

**focal parameter (\( p \)) of the parabola** The distance \( p \) from the focus \( F \) to the directrix \( d \) of the parabola.

**focus** One fixed point that determines the points of an parabola or one of two fixed points that determine the points of an ellipse/hyperbola.

**FOIL pattern** The method used to multiply two binomials in a single step. Find the sum of the products of the First terms, Outer terms, Inner terms, and Last terms.

For example, \( (x + 3)(2x - 5) = 2x^2 - 5x + 6x - 15 = 2x^2 + x - 15. \)

**formula** An equation/rule that uses letters to represent a relationship between two or more quantities.

**Four-Color Map Theorem** The four-color map theorem states that any map in a plane can be colored using only four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. A number of false proofs and false counterexamples have appeared since the first statement of the four-color theorem in 1840 (August Mobius, German mathematician). This theorem in field of graph theory was proven in 1976 by Kenneth Appel and Wolfgang Haken at the University Illinois. It was the first major mathematical theorem to be proved using a computer.

**Four-step process for Problem Solving** George Polya* (1887-1985), known as father of modern problem solving, created these famous four-step procedure:

1. **Understand the problem (SEE).** Discuss it. What is unknown? What are the data? What is the condition? Ask questions about it. Draw a picture of it. Restate it in your own words. Tell someone else about it.
2. **Device a plan (PLAN).** Have you seen it before? Do you know a theorem or formula that could be useful? Look at the unknown! Did you use all the data? Can you find pattern in the data?

3. **Carry out the plan (DO or solve equation or system of equations).** Carry out your plan. Carrying out your plan of the solution, check each step.

4. **Look back (CHECK).** Ensure you have used all the important information. Examine the solution obtained. Can you check and interpret your results? Decide whether or not the answer makes sense. Check that all of the given conditions of the problem are met by the answer. Put your answer in a complete sentence.

* “Learning begins with action and perception, proceeds hence to words and concepts, and should end in good mental habits.” (George Polya)

**fractal** A structure that is self-similar; each subdivision has the same structure as the whole and the structure of the object looks the same form any view. In other words, a fractal is a unique, enchanting geometric figure with an endless self-similarity property.

**fraction** A part of unity or several equal parts of unity, or, a number used to name part of a whole or group. The number (bellow fraction bar) which indicates how many parts a unit is divided into is called the denominator of the fraction; the number (above fraction bar) indicating how many parts are taken is the numerator of the fraction. For example, \( \frac{3}{7} \): here, 3 is the numerator and 7 is the denominator.

Famous Italian mathematician **Leonardo Pisano Fibonacci** (c. 1170-c 1250) in his book *Liber abaci* introduced us (c. 1202) to the bar we use in fractions, previous to this, the numerator has quotations around it.

**OPERATIONS ON FRACTIONS**

1. **Changing fraction to higher terms.**
   The value of a fraction is not changed if the numerator and denominator are multiplied by the same number. For example
   \[
   \frac{5}{7} = \frac{5 \cdot 2}{7 \cdot 2} = \frac{10}{14}
   \]
   This is called *changing the fraction to higher terms.*

2. **Reducing the fraction to the lowest terms.** The value of a fraction remains unchanged if the numerator and denominator are divided by one and the same number. This is called *reducing the fraction to lower terms.* The fraction/rational expression is in *lowest terms* if the greatest common factor (GCF) of its numerator and denominator is 1. To write a fraction/rational expression in lowest terms (or in simplest form) we will factor the numerator and denominator (1st step), and after that we will divide out (cancel out) any common factors (2nd step).

   **Example** Reduce the algebraic fraction/rational expression \( \frac{108a}{144a^2b} \) to the lowest terms.
Solution  We can see that 36a is the greatest common factor and we will cancel out it. Namely,
\[
\frac{108a}{144a^2b} = \frac{36a \cdot 3}{36a \cdot 4ab} = \frac{3}{4ab}
\]

3. Comparing Fractions

(a) If two fractions have the same numerator, the larger fraction is that with the smaller denominator.
For example, \(\frac{2}{11} > \frac{2}{15}\). If two fractions have the same denominator, the greater fraction is that with
greater numerator: \(\frac{7}{9} > \frac{5}{9}\). In order to compare two fractions with different denominators, either one
or both of the fractions have to be transformed so that the denominators are the same. This process of
changing fractions is called finding a common denominator, but this at the same time changing fractions
to higher terms. For example, \(\frac{5}{7} < \frac{8}{11}\), because \(\frac{5}{7} = \frac{55}{77}\), and \(\frac{8}{11} = \frac{56}{77}\).

Generally, \(\frac{a}{b} > \frac{c}{d}\) if \(ad > bc\).

(b) General method for finding a fraction between two given positive and nonequivalent fractions

If \(\frac{a}{b} < \frac{c}{d}\) then \(\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}\).

4. Adding and Subtracting Fractions

(a) If the denominators of the fractions are the same, the fractions may be added/subtracted by
adding/subtracting their numerators. This sum or difference will be the numerator of the answer.

Generally speaking, \(\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}\).

Example  Add and write answer in lowest terms (simplest form).
\[
\frac{1}{10} + \frac{3}{10} = \frac{4}{10} = \frac{2}{5}
\]

To add/subtract mixed numbers, separately find the sum/difference of the integral parts and fractional
parts. For example, \(3\frac{3}{5} + 5\frac{4}{5} = (3 + 5) + (\frac{3}{5} + \frac{4}{5}) = 8\frac{7}{5} = 9\frac{2}{5}\).

(b) If the denominators differ, first we have to transform fractions to a least common denominator
(LCD).

Example  Subtract and write answer in lowest terms.
\[
\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}
\]

(Answer is already in lowest terms); N. B. LCD (6, 4) = 12

5. Multiplication of Fractions

To multiply a fraction by a fraction, multiply the numerators together for the numerator of the product
and multiply the denominators together for the denominator of the product. Or generally speaking,
\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.
\]

If there are mixed numbers, convert them to improper fractions before multiplying. Also, before
multiplying, cancel out any common factors in the numerator and the denominator. For example,
\[
\frac{3}{10} \cdot \frac{5}{9} = \frac{3 \cdot 5}{10 \cdot 9} = \frac{1 \cdot 1}{2 \cdot 3} = \frac{1}{6}.
\]
6. Division of Fractions

To divide a fraction by a fraction, multiply the first fraction (the dividend) by the reciprocal of the second fraction (the divisor).

Example  Divide. Write the quotient in lowest terms.
\[ \frac{3}{4} \div \frac{7}{8} = (\text{N. B.: The reciprocal of } \frac{7}{8} \text{ is } \frac{8}{7}) = \frac{3 \cdot 8}{4 \cdot 7} = (\text{After canceling out common factor 4}) = \frac{3 \cdot 2}{1 \cdot 7} = \frac{6}{7} \]

“A man is like a fraction whose numerator is what he/she is and the denominator is what he/she thinks of himself/herself. The larger the denominator, the smaller the fraction.”

Leo N. Tolstoy (1828–1910)

fractional/rational expression  The ratio of two algebraic expressions.

free falling  An object that moves because of the action of gravity alone is said to be free falling. Galileo Galilei* (1564-1642) developed formula for freely falling objects that stated
\[ d = 16t^2, \]
where \( d \) is the distance in feet that an object falls in \( t \) seconds, regardless of weight.

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* Galileo Galilei: “All objects fall at the same rate in a vacuum”.

Free Will Theorem  The free will theorem of John H. Conway and Simon B. Kochen states that, if we have certain amount of free will, then, subject to certain assumptions so must some elementary particles. Conway and Kochen’s paper was published in Foundations of Physics in 2006 (Wikipedia). In other words, the theorem states that, if the two experiments in question are force to make choices about what measurements to take, the result of the measurements cannot be determined by anything previous to the experiments.

Frenet - Serret Formulas  These formulas describe the derivatives of the tangent (\( T \)), normal (\( N \)), and binormal (\( B \)) unit vectors in terms of each other. The formulas are named after the two French mathematicians who independently discovered them: Jean Frederick Frenet (1847), and Joseph Alfred Serret (1851). The Frenet – Serret formulas are:
\[
\frac{dT}{ds} = \kappa N \\
\frac{dN}{ds} = -\kappa T + \tau B
\]
\[
\frac{dB}{ds} = -\tau \kappa, \quad \text{where} \quad \frac{d}{ds} \text{ is the derivative with the respect to the arclength } s, \ \kappa \text{ is the curvature, and } \tau \text{ is the torsion the curve in space.}
\]

**frequency** How often something occurs. In simple harmonic motion, the frequency* is the number of cycles per unit of time.

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*Nikola Tesla (1856 – 1943): “Our entire biological system, the brain and the earth itself, work on the same frequencies.”

**frequency distribution** An ordered table of frequencies.

**friendly numbers** Two numbers are friendly/amicable if the sum of the factors of each equaled the other. 220 and 284 are the oldest pair known, and they date back to Pythagoras. A second pair: 17,296 and 18,416 were waiting whole two millennia and little bit more. Famous French mathematician Pierre de Fermat discovered this pair of friendly numbers at 1636. However, in the next three centuries has been discovered over 60 pairs of amicable numbers with one unbelievable fact: the second-lowest pair of all had been missed. In 1867 a 16-year-old Italian, Nicolo Paganini, demonstrated that 1,184 and 1,210 are friendly.

**front end rounding** We will use this procedure if we have to estimate answers especially in addition and subtraction two or more numbers. In front end rounding, each number is rounded to the highest possible place, so all the digits become 0 except the first digit.

**Example** Use front end rounding to estimate an answer. Then find the exact answer. John’s paycheck showed gross pay of $9,144. It also listed deductions of $3,922. What is his net pay after deductions?

**Solution** Estimate: $914 rounds to $900 (Next digit after 9 is 4 or less.), and $392 rounds to $400 (Next digit after 3 is 5 or more.)

We will use the rounding numbers and subtract to estimate John’s net pay.

\[
\$900 - \$400 = \$500
\]

Exact: We will use the original numbers and subtract to find the exact amount.

\[
\$914 - \$392 = \$522
\]

John’s paycheck will show the exact amount of $522. Because $522 is fairly close to the estimate of $500, John can quickly see that the amount shown on his paycheck probably is correct.

**function** A set of ordered pairs in which, for each value of the first component/coordinate/argument of ordered pairs, there is exactly one value of the second component/coordinate. In other words, this is a special type of relationship/correspondence between two sets, such that each member of the first set (the domain) corresponds to exactly one member of the second set (the range). It is possible for a function to have more than one input that yields the same output. Generally speaking, a function** is a relation between two or more variables. Leonhard Euler was the first to write \( f(x) \) to denote \( f \) (dependent variable) applied to the argument \( x \) (independent variable).

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*Ways of Representing Functions* A function can be expressed in four different ways.

1. **Verbal.** This is the first way in which functions are presented. For example, “the value of \( y \) depends on
the input value x plus 4”. In other words, we have here an linear function $y = x + 4$.

2. **Algebraic (analytical)**. This is most common, most concise, and most powerful representation (using equation of formula).

3. **Numerical**. This way can be done as a list of ordered pairs of numbers.

4. **Graphical**. In practice, we often use the graphical method of representing a function. This method turns out to be so convenient when it is rather difficult to represent a function analytically with its formula (equation).

**N. B.** One of the most important skills in algebra is converting a function between these four different ways of the same function.

**Johann Dirichlet** (1805-1859), German mathematician, proposed in 1837 the modern definition of function: “If a variable $y$ is so related to a variable $x$ that whenever a numerical value is assigned to $x$, there is a rule according to which a unique value of $y$ is determined, then $y$ is said to be a function of the independent variable $x$.”

**Fundamental Physical Constants**

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed of light in vacuum</td>
<td>c</td>
<td>$2.99792458 \times 10^8 \frac{m}{s}$</td>
</tr>
<tr>
<td>Planck constant</td>
<td>h</td>
<td>$6.6260755 \times 10^{-34} J \cdot s$</td>
</tr>
<tr>
<td>Gravitation constant</td>
<td>G</td>
<td>$6.67259 \times 10^{-11} m^3 \cdot kg^{-1} \cdot s^{-2}$</td>
</tr>
<tr>
<td>Boltzmann constant</td>
<td>k</td>
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</tr>
<tr>
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</tr>
<tr>
<td>Coulomb constant</td>
<td>$k_e$</td>
<td>$8.967552 \times 10^9 N \cdot m^2 \cdot C^{-2}$</td>
</tr>
<tr>
<td>Faraday constant</td>
<td>F</td>
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</tr>
<tr>
<td>Mass of electron</td>
<td>$m_e$</td>
<td>$9.1093897 \times 10^{-31} \text{ kg}$</td>
</tr>
<tr>
<td>Mass of proton</td>
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<tr>
<td>Stefan-Boltzmann constant</td>
<td>$\sigma$</td>
<td>$5.67051 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$</td>
</tr>
</tbody>
</table>
Fundamental Principle of Counting (The multiplication rule) If $n$ independent events occur, with $m_1$ ways for the first event to occur, $m_2$ ways for the second event to occur, 
:
and $m_n$ ways for the nth event to occur,
then there are $m_1 \cdot m_2 \cdot m_3 \cdot \cdots \cdot m_{n-1} \cdot m_n$ different ways for all $n$ events to occur.

**Example 1** How many 7-digit telephone numbers are possible if the first digit cannot be 0 and 
(a) only odd digits may be used? 
(b) the telephone number must be a multiple of 10 (that is, it must end in 0)? 
(c) the telephone number must be a multiple of 100? 
(d) the first 3 digits are 763? 
(e) no repetition are allowed? 

**Solution** 
(a) Because we have 5 single odd digits (1, 3, 5, 7, 9) there are 5 choices for each of the seven telephone digits. So, the number of 7-digit telephone numbers with only odd digits is $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 78,125$. 
(b) The first digit can one of 9 choices, because 0 is excluded. For next six digits we have 10 choices for each, and for the last digit can be 0 only (one choice). So, our number is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 1 \cdot 1 = 900,000$. 
(c) Because the first digit cannot be 0, there are 9 choices for that digit. For next five digits we have 10 choices for each, and for the last two digit only one choice for each. So, our number is $9 \cdot 10 \cdot 10 \cdot 10 \cdot 1 \cdot 1 \cdot 1 = 90,000$. 
(d) The first three digits each have only one choice, and for other five digits we have 10 choices for each. So, our number for this case is $1 \cdot 1 \cdot 1 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10,000$. 
(e) The first digit can be one of 9 choices, because 0 is excluded, and for the second digit we have 9 choices too, because we used already one digit for the first place. For the third digit we have 8 choices, ..., and for the last digit we have 4 choices. So, our number is $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 544,320$. 

**Example 2** How many four-digit numbers can be composed of digits 1, 5, 6, and 7? 

**Solution** There are exactly four possibilities for every digit of a four-digit number. Consequently, according to the multiplication rule, we can have $4 \cdot 4 \cdot 4 \cdot 4 = 256$ four-digit numbers. 

**Example 3** How many two-digit numbers are there both of whose digits are even? 

**Solution** One of the digits 2, 4, 6, and 8 can be the tens digit of the required numbers (4 possibilities) and one of the digits 0, 2, 4, 6 and 8 can be the units digit (5 possibilities). Thus there are $4 \cdot 5 = 20$ required numbers.

Fundamental Property of Fractions If $a$, $b$, and $k$ are real numbers, with $b, k \neq 0$, then
\[
\frac{ka}{kb} = \frac{a}{b}.
\]
fundamental rectangle  *(See conic sections)* The fundamental rectangle *(with $l = 2a$ and $w = 2b$)* is used as a guide in sketching the graph of a hyperbola. The extended diagonals of this rectangle are asymptotes of the hyperbola.

**Fundamental Theorem of Algebra** Every function defined by a polynomial of degree 1 or more has at least one complex zero. Generally speaking, polynomial of $n$th degree has at most $n$ distinct zeros/roots. This theorem was first proved by famous German mathematician Carl Friedrich Gauss *(1777-1855)* in his doctoral thesis in 1799.

**Fundamental Theorem of Arithmetic (or the unique-prime-factorization theorem)** Every composite number can be expressed as a unique product of prime factors (prime numbers). For example, $120 = 2^3 \cdot 3 \cdot 5$.

**Fundamental Theorem of Calculus** See Newton-Leibnitz Formula.

**Fundamental Theorem of Linear Programming** If an optimal value for a linear programming problem exists, it occurs at a vertex of the region of feasible solutions.

**G**

game theory  A study of strategic decision making. This is a branch of mathematics that seeks to understand why an individual makes a particular decision and how the decisions made by an individual affects others.

gamification  The gamification is the application of game theory concepts techniques to non-game activities. In other words, the gamification is the use of game mechanics and game design techniques in non-game contexts.

grounding figure  A geometric figure is a point, line segment, line, plane, solid, or combination of these.

general term (nth term) of a sequence  In the sequence $a_1, a_2, a_3, \ldots, a_n$ the general term (or nth term) is $a_n$.

gemetric mean  The geometric mean of $n$ quantities is the $n$th root of their product. For example, geometric mean of $2, 4,$ and $8$ is $\sqrt[3]{2 \cdot 4 \cdot 8} = \sqrt[3]{64} = 4$.

*The geometric mean is always less than the arithmetic mean except for the case when all numbers are equal.*

gemetric sequence/progression*  A sequence of numbers $(a_1, a_2, a_3, \ldots, a_n)$ such that the ratio of each term to the immediately preceding one is a constant called the common ratio $(r)$. In this sequence each term after the first $(a_1)$ is obtained by multiplying the preceding term by fixed nonzero real
number, called the common ratio.

*Formula for general (nth) term: \( a_n = a_1 r^{n-1} \).

Formula for the sum of the first \( n \) terms of a geometric sequence: \( S_n = \frac{a_1(1 - r^n)}{1-r} \).

**geometric series** A series for which the associated sequence is geometric.

**Geometry** A branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids. The name stemming from Greek term “geos metreo” means “earth measurement”. Briefly, geometry is the study of form and shape.

*Plato (427 BC – 347 BC), philosopher in Classical Greece and the founder of the Academy in Athens, the first institution of higher learning in the Western world: “Geometry existed before creation.”

**Goldbach’s conjecture** In 1742, Christian Goldbach, German mathematician (1690 – 1764), conjectured in a letter to Leonhard Euler that every even number greater than or equal to 4 can be represented as the sum of two (not necessarily distinct) prime numbers (For example 10 = 3 + 7). This conjecture is one of the oldest and best-known unsolved problems in number theory and in all of mathematics.

**golden ratio** \( \varphi \) The quantities \( a + b \) and \( a \) are said to be in the golden ratio (golden section), if the whole segment \( (a + b) \) is to the larger part \( (a) \) as the larger part \( (a) \) is to the smaller part \( (b) \), i.e. if \( \frac{a+b}{a} = \frac{a}{b} \). The numerical value of this ratio is denoted by the Greek letter \( \varphi \) (phi). \( \varphi \) is an irrational number (1.61803...) whose decimal expansion never ends and never repeats. If we let \( a + b = 1 \) (the unit segment), then \( \varphi = \frac{1}{a} = \frac{a}{1-a} \Rightarrow a^2 + a - 1 = 0 \Rightarrow a \approx 0.61803... \Rightarrow \varphi = 1.61803... \). Therefore \( \varphi \) is also an algebraic number. Generally, two quantities are in the golden ratio \( \varphi \) if their ratio is the same as the ratio of their sum to the larger of two quantities. The golden number \( \varphi \) first appeared in Euclid’s Elements, written around 350 BC.

**golden rectangle** A rectangle in which the length \( l \) and the width \( w \) satisfy the proportion of the golden ratio: \( \frac{w}{l-w} = \frac{l}{w} \); \( w > l - w \).

**gradient vector** Let \( f(x, y) \) be differentiable at a point \( P(x, y) \) and let \( f(x, y) \) have partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \). Then the gradient or the gradient vector of \( f(x, y) \), denoted by \( \nabla f \), at a point \( P(x, y) \) is a vector given by

\[
\nabla f(x, y) = \frac{\partial f(x,y)}{\partial x} i + \frac{\partial f(x,y)}{\partial y} j = f_x(x, y) i + f_y(x, y) j
\]

**graph of an equation** The set of all points that correspond to all of the ordered pairs that satisfy the equation. In other words, the graph is a line, curve, or collection of points that represents all the solutions of an equation. Generally speaking, the graph is a diagram that shows a relationship among numbers.
graphing a rational function  Let \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomials without common factors. To sketch the graph of the rational function \( f(x) \), follow these steps.

1. Determine the domain of \( f(x) \) and find any vertical asymptotes. You will find any vertical asymptotes by setting the denominator \( q(x) \) equal to 0 and solving corresponding equation for \( x \). Solutions of this equation will be equations of vertical asymptotes. For example, for the rational function \( f(x) = \frac{x+1}{(x+2)(x-3)} \) you will set the denominator equal to 0 and solve it.

\[ (x + 2)(x - 3) = 0 \rightarrow x + 2 = 0 \text{ or } x - 3 = 0. \]

So, \( x = -2 \) and \( x = 3 \) will be equations of the vertical asymptotes and for \(-2\) and \(3\) the function \( f(x) \) is not defined. Therefore, the domain of \( f(x) \) is \(( -\infty, -2) \cup (-2, 3) \cup (3, \infty)\) or set of all real numbers, except \(-2\) and \(3\).

2. Find any horizontal or oblique asymptotes.
   (a) If the numerator \( p(x) \) has the lower degree than the denominator, then there is a horizontal asymptote \( y = 0 \) (equation of the x-axis).
   (b) If the numerator and denominator have the same degree, the equation of the horizontal asymptote will have equation \( y = \text{quotient of the leading coefficients of } p(x) \text{ and } q(x) \).

For example, \( f(x) = \frac{2x^2+1}{3x^2+2x-1} \). The equation of the horizontal asymptote will be \( y = \frac{2}{3} \).
   (c) If the numerator is of degree exactly one more than the denominator, then there will be an oblique/slanted asymptote. To find it, divide the numerator by the denominator and disregard the remainder. The equation of the oblique asymptote will be \( y = \text{the quotient} \). For example, \( f(x) = \frac{x^2+1}{x-2} \), after division we will have \( f(x) = \frac{x + 2}{x - 2} \), and equation of the oblique asymptote will be \( y = x + 2 \).

3. Find the \( y \)-intercept by evaluating \( f(0) \).
4. Find the \( x \)-intercepts, if any, by solving \( f(x) = 0 \). Note, \( f(x) = 0 \rightarrow p(x) = 0 \).
5. Determine whether the graph will intersect its nonvertical asymptote \( y = b \) or \( y = mx + b \) by solving equation \( f(x) = b \) or \( f(x) = mx + b \).
6. Plot at least one point both \( \text{between and beyond} \) each \( x \)-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptote.

graphing techniques  In comparison with the graph of \( y = f(x) \):

1. The graph of \( y = f(x) + 3 \) is translated 3 units up (vertical translation/shift).
2. The graph of \( y = f(x) - 3 \) is translated 3 units down.
3. The graph of \( y = f(x + 3) \) is translated 3 units to the left (horizontal translation/shift).
4. The graph of \( y = f(x - 3) \) is translated 3 units to the right.
5. The graph of \( y = 3f(x) \) is a vertical stretching of the graph \( y = f(x) \) by a factor of 3.
6. The graph of \( y = \frac{1}{3} f(x) \) is a vertical shrinking/compressing of the graph \( y = f(x) \) by a factor of 3.
7. The graph of \( y = f(3x) \) is a horizontal shrinking/compressing of the graph \( y = f(x) \).
8. The graph of \( y = f(\frac{1}{3} x) \) is a horizontal stretching of the graph of \( y = f(x) \).
9. The graph of \( y = -f(x) \) is reflected across the x-axis.
10. The graph of \( y = f(-x) \) is reflected across the y-axis.

**Gravity** The gravity is the force that causes two particles to pull towards each other. Isaac Newton* defined the gravity** as a force – one that attracts all objects to all other objects. Albert Einstein said gravity is a result of the curvature of space-time, but he did not explain this notion – the curvature of space-time.

………………
*Isaac Newton: “Gravity explains the motions of the planets, but it cannot explain who set the planets in motion. God governs all things and knows all that is or can be done.”

**Shouryya Ray**, an Indian-born teenager (16), who lives in Dresden, eastern Germany, has become the first person to solve a mathematical riddle posed by Sir Isaac Newton more than 300 years ago. He worked out (May, 2012) how to calculate exactly the path of a projectile under gravity and subject to air resistance while working on a school project.

Stephen Hawking, English theoretical physicist: “Because there is a law such as gravity, the universe can and will create itself from nothing.”

**Great Circle** A circle on a sphere that divides the sphere into two equal parts.

**Greatest Common Divisor (GCD)** The largest factor that two or more numbers(expressions) have in common. For example, GCD (12, 20) = 4.

**Greatest Common Factor (GCF)** The largest factor that is a factor of the given numbers(expressions). For example, GCF (10\(x^2\), 25\(x^3\)\(y\)) = 5\(x^2\).

**Green’s* Theorem**
1. **Flux-Divergence or Normal Form** Let \( C \) be piecewise smooth, simple closed curve enclosing a region \( D \) in the plane. Let \( \mathbf{F} = M \mathbf{i} + N \mathbf{j} \) be a vector field with \( M \) and \( N \) having continuous first partial derivatives in an open region containing \( D \). Then the outward flux of \( \mathbf{F} \) across \( C \) equals the double integral of div \( \mathbf{F} \) over the region \( D \) enclosed by the curve \( C \).
\[
\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_D \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dxdy
\]
2. **Circulation-Curl or Tangential Form** For the vector field \( \mathbf{F} \) the counterclockwise circulation of \( \mathbf{F} \) around \( C \) equals the double integral of (curl \( \mathbf{F} \) ) \( \cdot \mathbf{k} \) over \( D \).
\[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dxdy
\]
3. **Calculating Area with Green’s Theorem** If a simple closed curve \( C \) in the plane and the region \( D \) it encloses satisfy the hypotheses of Green’s Theorem, the area of \( D \) is given by
\[
\text{Area of } D = \frac{1}{2} \oint_C (xdy - ydx)
\]
………………
*George Green (1793 – 1841), British mathematical physicist, who first indicates relationship between the line integral around \( C \) and a double integral over region \( D \) (consisting of \( C \) and its interior), and he was the first person to create a rudimentary mathematical theory of electricity and magnetism. Green's
life story is remarkable in that he was almost entirely self-taught. He received only about one year of formal schooling as a child, between the ages of 8 and 9.

**gross pay** Total wage before deductions are made.

**group** A finite or infinite set of elements together with a binary operation (called group operation) that together satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. The concept of a group is one of the most fundamental concepts of modern (higher) algebra with numerous applications in almost every branch of mathematics and other sciences.

**grouping symbols** The symbols, such as parentheses ( ), brackets [ ], braces { }, or the fraction bar that indicate that the operations within them should be done first. When an expression contains more than one grouping symbol, the computations in the innermost grouping symbols should be done first.

**Example** Solve.

3\{ 2\left[ 4 \left( 2(3-x) \right) -1 \right] -1 \} = 4\left[ 2(4x-3) + 7 \right] - 25

**Solution** Remove parentheses by using the distributive laws. Do all calculations within grouping symbols before operations outside with hierarchy { ( ) }.

3\{2\left[ 4 - 6\left( 2x \right) -1 \right] -1 \} = 4\left[ 8x - 6 + 7 \right] - 25

Collect like terms

3\{2\left[ 2x - 2 \right] -1 \} = 4\left[ 8x + 1 \right] - 25

Remove brackets

3\{4x - 4 \} = 32x + 4 - 25

Collect like terms

3\{4x - 5 \} = 32x - 21

Remove braces

12x - 15 = 32x - 21

Group terms with x by adding – 32x

12x - 15 - 32x = 32x - 21 - 32x

Collect like terms

- 20x - 15 = - 21

Add 15 to isolate – 20x

- 20x - 15 + 15 = - 21 + 15

Combine like terms

- 20x = - 6

Isolate x by dividing by -20 and simplifying

\[ \frac{-20x}{-20} = \frac{-6}{-20} \quad \text{or} \quad x = \frac{3}{10} \]

**half-life** The amount of time necessary for half of a the radioactive quantity to decay.

**Example** A sample of 500 g of radioactive lead 210 decays to polonium 210 according to the exponential function defined by \( A(t) = 500e^{-0.032t} \), where \( t \) is time in years. Find the half-life.

**Solution** We have to find \( t \) when \( A(t) = 250 \) g.

Therefore, \( 250 = 500e^{-0.032t} \) → \( 0.5 = e^{-0.032t} \) /ln (Take natural logarithms on both sides) → \( \ln 0.5 = -0.032t\ln e \) → (NB: \( \ln e = 1 \)) → \( t = \frac{\ln 0.5}{-0.032} \approx 21.66. \)

The half-life is about 21.66 yr.
**half-open interval** The set of all real numbers \( x \) such that \( a \leq x < b \) is termed a half-open interval and is denoted by \([a, b)\). The interval \((a, b]\) is defined in a similar manner, that is, the set of numbers \( x \) satisfying the inequality \( a < x \leq b \).

**harmonic mean** The harmonic mean of numbers (quantities) \( a \) and \( b \) is
\[
1\div \frac{1}{a} + \frac{1}{b} = \frac{2ab}{a+b}.
\]
The harmonic mean between two quantities does not exceed the arithmetic mean of the quantities.

* Marcus Aurelius, Roman emperor (121 - 180): “He who lives in harmony with himself lives in harmony with the universe.”

**harmonic motion** The (simple) harmonic motion is a type of a periodic/oscillatory motion in which movements are symmetrical about region of equilibrium. The position of a point oscillating about an equilibrium (or rest) position at time \( t \) is modeled by equations
\[
s(t) = a \sin \omega t \quad \text{or} \quad s(t) = a \cos \omega t,
\]
where \( a \) and \( \omega \) are constants, with \( \omega > 0 \). The amplitude of the motion is \( |a| \), the period is \( \frac{2\pi}{\omega} \), and the frequency is \( \frac{\omega}{2\pi} \) oscillations per time unit.

**harmonic series** A series of the form
\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots
\]

**Hawking’s equation** Stephen Hawking (1942 – 2018), famous British physicist and explorer of black holes, left us so rare equation about entropy and black holes:
\[
S = \frac{\pi Ak c^3}{2hG}, \quad \text{where the letters} \ S \text{ and} \ A \text{ represent entropy and black hole’s area,} \ h \text{ is Plank’s constant,} \ G \text{ is Newton’s universal gravitational constant, and} \ k \text{ is the Boltzmann’s constant.}
\]

**head-to-tail method** In vector addition, finding the sum of two vectors by placing the tail of one vector at the head of the other; the vector drawn from the tail of the first vector to the head of the second is the vector sum.

**height of a polygon** The length of an altitude of a polygon.

**Height of a Projected Object** When an object is projected upward and released at some initial velocity \( v_0 \) (feet per second) from an initial height \( s_0 \), it is moving in the opposite direction of the force of gravity. Since our convention states that the direction of gravity is positive, the upward initial velocity is then a negative number. The object moves upward, slowing down from its initial velocity until it reaches its maximum height. Then the object falls toward the ground. For this consideration the air resistance is negligible. The height \( s \) (in feet) of the object projected directly upward is given by the projectile height function (or displacement function):
\[
s(t) = 16t^2 + v_0 t + s_0, \quad \text{where} \ t \text{ is the number of seconds after the object is projected.}
\]
Heron’s* (area) formula  If a triangle $ABC$ has sides of lengths $a$, $b$, and $c$, which semi-perimeter $s = \frac{1}{2}(a + b + c)$, then the area of the triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$.  

*The formula is credited to Heron of Alexandria, Greek scholar, who lived in 1st century A.D., and a proof can be found in his famous book *Metrica*, written c. A.D 60. It has be suggested that Archimedes of Syracuse (c. 287 BC – c. 212 BC), Greek mathematician and inventor, knew the formula two centuries earlier.

Heron triangle  A Heron triangle is a triangle whose sides and area are integers.

hexagon  A polygon with six sides.

Hilbert* space  A metric space that is linear and complete and (usually) indefinite-dimensional. 

*David Hilbert (1862 – 1943)

histogram  A bar graph of grouped data.

Homeowner’s Mortgage (See mortgage)  A homeowner’s mortgage is a long-term loan in which the property is pledged as security for payment of the difference between the down payment and the sale price.

homogeneous differential equation  A homogeneous differential equation is an equation of the form $P(x, y)dx + Q(x, y)dy = 0$, where $P$ and $Q$ are homogeneous function of the same degree. To solve an equation of this form by the method of separation of variables, we will use the substitution $y = ux$, and after differentiation we will have $\frac{dy}{dx} = x \frac{du}{dx} + u$ or $dy = xdu + udx$.

homogeneous differential equation  A first order ordinary differential equation in the form $M(x,y)dx + N(x,y)dy = 0$ is homogeneous type if both functions $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree $n$.

homogeneous function  A function $f(x)$ is homogeneous of degree $n$ if $f(\alpha x) = \alpha^n f(x)$, where $\alpha$ is a constant parameter.

For example, the function $f(x, y) = x^3 + 4x^2y + 3y^3$ is homogeneous function of degree 3 because $f(\alpha x, \alpha y) = (\alpha x)^3 + 4(\alpha x)^2(\alpha y) + 3(\alpha y)^3 = \alpha^3 x^3 + 4\alpha^2 x^2 ay + 3\alpha^3 y^3 = \alpha^3 (x^3 + 4x^2 y + 3y^3) = \alpha^3 (x^3 + 4x^2 y + 3y^3) = \alpha^3 f(x,y)$.

Hooke’s* Law  This law states that the force $F$ required to compress or stretch a spring $x$ units from its natural length is proportional to $x$. That is, $F(x) = kx$

where the constant of proportionality $k$ (the spring constant) depends on the specific nature of the
spring.

* Robert Hooke (1635 – 1703), English natural philosopher, architect and polymath.

**horizontal asymptote** A horizontal line that graph approaches as \(|x|\) increases without bound is a *horizontal asymptote*. The line \(y = b\) is a horizontal asymptote of the graph \(y = f(x)\) if \(y \to b\) as \(|x| \to \infty\).

**horizontal component** When a vector \(u\) is expressed as an ordered pair in the form \(u = \langle a, b \rangle\), the number \(a\) is the horizontal component of the vector.

**horizontal line** A straight line with zero slope.

**horizontal-line test** If any horizontal straight line intersects the graph of a function in no more than one point, then the function is one-to-one. In other words, any horizontal line will intersect the graph of a one-to-one function in at most one point.

**hyperbola** (See ‘conic sections’) The set of all points in a plane such that the absolute value of the difference of distances from two fixed points in the same plane is constant. The two fixed points are called the *foci* of the hyperbola, and the distance between them is the *focal length*.

**hyperbolic functions** The hyperbolic functions are special case of the exponential function and they are formed by taking combinations of the two exponential inverses function \(e^x\) and \(e^{-x}\). There are six hyperbolic functions*: \(\sinh x\), \(\cosh x\), \(\csch x\), \(\sech x\), \(\tanh x\), \(\coth x\).

**Definitions of hyperbolic functions**

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} ; \\
\cosh x &= \frac{e^x + e^{-x}}{2} ; \\
\csch x &= \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} ; \\
\sech x &= \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} ; \\
\tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} ; \\
\coth x &= \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} .
\end{align*}
\]

**Basic identity**: \(\sinh^2 x - \cosh^2 x = 1\)

**hypotenuse** The longest side in a right triangle. Two shorter sides in a right triangle are so called *legs*.

**hypothesis** The phrase following the word *if* in in conditional statement. Actually, when a scientist or mathematician makes a prediction based on specific observations, it is called a hypothesis or conjecture.

...
all numerical values of x and y the right and left sides yield the same number. Solution set is the set of all real numbers.

**Identity equation** If solving a linear equation leads to a true statement such as 0 = 0, the equation is an identity. Its solution set is a set of all real numbers.

**Identity property of 0** The statement that the sum of a number and 0 is always the original number, i.e., \( a + 0 = 0 + a = a \).

**Identity property of 1** The statement that the product of a number and 1 is always the original number, i.e., \( a \cdot 1 = 1 \cdot a = 1 \).

**if-then statement** A conditional statement made up of two parts: “If . . . , then . . . .” The first statement following if, is called the hypothesis or antecedent. The second statement, following then, is called the conclusion or consequent.

**image** A shape (or set) that results from a transformation of figure (or set) known as the preimage.

**imaginary numbers** Imaginary numbers are numbers of the form \( bi \) where \( b \) is non-zero real number and \( i \) is the imaginary unit. For example, \( 5i \) is an imaginary number.

**imaginary part** In the complex number \( a + bi \), \( b \) is called imaginary part of the complex number.

**imaginary unit** \( i \) The imaginary unit’s core property is that \( i^2 = -1 \). There are in fact two square roots of \(-1\), namely \( i \) and \(-i\). Just as there are two square roots of every other real number, except zero, which has one square root. In other words, the equation \( i^2 = -1 \) is the only definition of the imaginary unit \( i \). Famous Swiss mathematician **Leonhard Euler** (1707–1783) introduced notation of \( i \) to represent \( √{-1} \), which he called an “imaginary number.” Imaginary numbers, despite their name, they are not really imaginary at all.

**improper fraction** A fraction that has a numerator greater than or equal to the denominator.

* To change an improper fraction into a mixed number, divide the numerator by the denominator. The quotient is the whole number part of the mixed number. The remainder is the new numerator. For example, \( \frac{18}{11} \) is 1 with remainder 7, or \( 1 \frac{7}{11} \).

**improper integral** Integrals with infinite limits are improper integrals of Type I. Integrals of functions (integrand) that become infinite at a point within the interval of integration \([a, b]\) are improper integrals of Type II. In other words, an improper integral is actually the limit of a definite integral as an endpoint of the interval of integration approaches either a specified real number or \( ±\infty \) or, in some cases, as both endpoints approach limits \( \lim_{a \to -\infty} \int_a^b f(x)dx, \lim_{b \to \infty} \int_a^b f(x)dx \), etc.). Integrals are also
improper if the integrand is undefined at an (or more) interior point of the interval of integration \( \int_{1}^{5} \frac{e^x}{\sqrt{x-3}} \, dx \), etc.

**inclusive OR** Indicating either one or the other or both.

**inconsistent system of equations** A system of equations for which there is no solution, or the solution set is empty.

**increasing function** A function \( y = f(x) \) is increasing on an interval \( I \) if, whenever \( a < b \) in \( I \), \( f(a) < f(b) \).

**indefinite integral** \( \int \) * The collection of all antiderivatives of a given function \( f(x) \) is called the *indefinite integral* of \( f(x) \) and is denoted by the symbol \( \int f(x) \, dx \).

*German mathematician and philosopher **Gottfried Wilhelm von Leibniz** (1646-1716) introduces several notations used to this day, for instance, the integral sign \( \int \) representing an elongated S, from Latin word *summa*.

**independent equations** The equations that are not dependent.

**independent events** Event \( A \) and event \( B \) are independent events if the occurrence of either event in no way affects the probability of occurrence of the other event.

**independent variable** If the value of the variable \( y \) depends on the value of the variable \( x \), then \( x \) is called the independent variable.

**indeterminate/undefined form** *(See ‘The Limit of a Function at a Point’) A expression if it is not definitively or precisely determined. These expressions have no preassigned value; they can be evaluated only through a limiting process. There are seven *indeterminate forms*: \( 0^0, \frac{0}{0}, \frac{\infty}{\infty}, 1^\infty, \infty^0, \infty - \infty, 0 \cdot \infty \). An indeterminate or undefined form does not mean that the limit is non-existent or cannot be determined, but rather that properties of its limits are not valid. In these cases, a particular operation(s) can be performed to solve (determine/define) each of the indeterminate forms. There are many methods of evaluating indeterminate forms (factoring, division of the numerator and denominator by highest power of the variable, differentiation, ...). To say, for example, \( \frac{0}{0} \) is an indeterminate form does not mean simply that \( \frac{0}{0} \) by itself does not represent any number.

**index of the root** *(See ‘Finding the root of a number’) In the radical \( \sqrt[n]{x} \) (nth root) the number \( n \) is called the *index of the root.*
**Indicator/key words** Certain key words* indicate/suggest certain mathematical operations. If you have to write the phrase as a variable expression or equation use $x$ to represent “a number.”

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<thead>
<tr>
<th>KEY WORDS</th>
<th>ALGEBRAIC EXPRESSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>A number plus 4</td>
<td>$x + 4 \quad \text{or} \quad 4 + x$</td>
</tr>
<tr>
<td>The sum/total of 5 and a number</td>
<td>$5 + x \quad \text{or} \quad x + 5$</td>
</tr>
<tr>
<td>7 more than a number</td>
<td>$7 + x \quad \text{or} \quad x + 7$</td>
</tr>
<tr>
<td>11 added to a number</td>
<td>$11 + x \quad \text{or} \quad x + 11$</td>
</tr>
<tr>
<td>A number increased by 8</td>
<td>$x + 8 \quad \text{or} \quad 8 + x$</td>
</tr>
<tr>
<td>10 less than a number</td>
<td>$x - 10$</td>
</tr>
<tr>
<td>A number less 20</td>
<td>$x - 20$</td>
</tr>
<tr>
<td>A number subtracted from 4</td>
<td>$4 - x$</td>
</tr>
<tr>
<td>5 subtracted from a number</td>
<td>$x - 5$</td>
</tr>
<tr>
<td>A number decreased by 9</td>
<td>$x - 9$</td>
</tr>
<tr>
<td>53 decreased by 11 times a number</td>
<td>$53 - 11x$</td>
</tr>
<tr>
<td>10 minus a number</td>
<td>$10 - x$</td>
</tr>
<tr>
<td>The difference of twice a number, and 5</td>
<td>$2x - 5$</td>
</tr>
<tr>
<td>7 times a number</td>
<td>$7x$</td>
</tr>
<tr>
<td>$\frac{3}{5}$ of a number</td>
<td>$\frac{3}{5}x$</td>
</tr>
<tr>
<td>The product of 9 and a number</td>
<td>$9x$</td>
</tr>
<tr>
<td>Double a number (meaning “2 times”)</td>
<td>$2x$</td>
</tr>
<tr>
<td>The sum of 15 and a number added</td>
<td>$7x + (15 + x)$</td>
</tr>
</tbody>
</table>
The quotient of $-7$ and a number $\frac{-7}{x}$

A number divided by 11 $\frac{x}{11}$ or $x \div 11$

10 subtracted from 3 times a number $3x - 10$

The result is $= \ldots$ 

**Example 1** Twice the difference of a number and 1 is 4 more than that number.

$2(x - 1) = x + 4$

To solve this equation you will remove parentheses and get all the $x$-terms on one side, etc.

$2x - 2 = x + 4 \Rightarrow 2x - x = 4 + 2 \Rightarrow x = 6$

**Example 2** One number is 3 less than another number. If the sum of the two numbers is 177, find each number.

We will let $x =$ one number $\Rightarrow x - 3 =$ another number.

Therefore, $x + (x - 3) = 177 \Rightarrow x + x - 3 = 177 \Rightarrow 2x = 180 \Rightarrow x = 90 \Rightarrow x - 3 = 87$.

So, the solution set is $\{87, 90\}$.

**Indirect Proof or Proof by Contradiction** A proof in which the statement that you want to prove is assumed to be false and a contradiction or other “absurdity” is shown to follow from the assumption. This form of argument is known by its Latin name *reduction ad absurdum*.

**Inductive Reasoning** The process of reasoning to a general conclusion through observations of specific cases. In other words, inductive reasoning is the logical application of specific observations to make a generalization. In science it is used to formulate testable hypothesis/conjectures.

**Inequality** Two expressions connected by an inequality symbol ($<, >, \leq, \geq, \text{ or } \neq$).

* *Aristotle* (384 BC – 322 BC): “The worst form of inequality is to try to make unequal things equal.”

**Inertia** The tendency of an object in motion to remain in motion, or an object at rest, unless acted upon by a force.

**“Inexpressible” Integrals** If an integral is not expressed in terms of elementary functions, we then say that it “cannot be found”, or that is “inexpressible”. For instance, the functions $e^{-2x^3}$, $\frac{1}{lnx}$, $\frac{sinx}{x}$, $\sqrt{1 + x^5}$ have antiderivatives but are not elementary functions. Therefore, the integrals $\int e^{-2x^3} \, dx$, $\int \frac{1}{lnx} \, dx$, $\int \frac{sinx}{x} \, dx$, $\int \sqrt{1 + x^5} \, dx$ are said to be “inexpressible”.

**Inferential Statistics** Conclusions are made from analysis of a sample.
infinity* A quantity without end/bound. Metaphysically speaking, infinity is only one of attributes of eternity that does not reveal the substance and essence of eternity.

* George Cantor (1845-1918): “A collection is infinite if some of its parts are as big as the whole.”

infinite sequence An sequence if its domain is the set all natural numbers.

infinite series An expression of the form \( S = a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{i=1}^{\infty} a_i \).

infinite set An set that has an unending list of distinct elements/members.

infinite sequence A sequence \( \{ a_n \} \) is called infinitely large if \( \lim_{n \to \infty} a_n = \infty \).

infinite small sequences A sequence \( \{ a_n \} \) is called infinitely small (infinitesimal) if its limit is equal to zero, i.e. \( \lim_{n \to \infty} a_n = 0 \).

infinitesimal A variable whose limit is zero is termed an infinitely small quantity or an infinitesimal. In other words, this is an incalculably small quantity or a quantity less than any finite quantity yet not zero. English mathematician John Wallis (1616 – 1703) used \( \frac{1}{\infty} \) for an infinitesimal.

inflection point A point where a graph changes concavity is called an inflection point.

N. B. At an inflection point for a function \( f \), the second derivative \( f'' \) is 0 or does not exist (See “sufficient condition for inflection”).

initial side When a ray is rotated around its endpoint to form an angle, the ray in its initial position is called initial side/leg of the angle.

Initial Value Problem (IVP) The problem of finding a function \( y(x) \) when we know its derivative \( y'(x) \) and its value \( y_0 \) at a point \( x = x_0 \) is called initial value problem. This problem can be solved in two steps.

1. \( \frac{dy}{dx} = f(x) \implies \int dy = \int f(x) \, dx \implies y = F(x) + C \) (the general solution)

2. Using the initial data \( y(x_0) = y_0 \), plug it into the general solution and solve for \( C \) to get the particular solution.
Example  Solve the initial value problem \( \frac{dy}{dx} = x - 5, \ y(0) = -2. \)

**Solution**  Step 1: \( \frac{dy}{dx} = x - 5 \Rightarrow dy = (x - 5)dx \Rightarrow \int dy = \int (x - 5)dx \Rightarrow y = x^2 - 5x + C; \)

Step 2: When \( x = 0, y = -2 \) or \( y(0) = -2 \Rightarrow -2 = 0 - 5 \cdot 0 + C \Rightarrow C = -2. \)

So, the particular solution is \( y = x^2 - 5x - 2. \)

**injection**  A function that preserves distinctness: it never maps distinct elements of its domain to the same element of its range. In other words, every output (y) has at most one input (x). An injective function is also said to be a one-to-one function (N. B. don’t be confused with one-to-one correspondence, i.e. a bijective function).

**input**  An element of the domain of a function (an x-value).

**Integers***  The set of all whole numbers and their opposites: \{..., -3, -2, -1, 0, 1, 2, 3, ...\}. Three dots on each side indicate that list continues in both directions without end. Greek Diophantus of Alexandria who lived in the 3rd century is generally credited with being “father” of algebra as we know it today. He was probably the first to make a clear distinction between positive and negative numbers. He used names that meant “forthcoming” (positive numbers) and a “wanting” (negative numbers).

*Leopold Kronecker* (1823 - 1891): “*God made integers, all else is the work of man.*”

**Integral**  An integral is a mathematical object that can be interpreted as area or generalization of area, bounded by the graph of the function, x axis and limits of the integral. Integrals, together with derivatives, are the fundamental notions of mathematical analysis (calculus).

**Integral equation**  An equation involving a function \( f(x) \) and integrals of that function to solved for \( f(x) \).

The most basic type of integral equation is a Fredholm’s* equation of the first type

\[ f(x) = \int_a^b K(x, t)\varphi(t) \, dt, \]

where \( \varphi(t) \) is an unknown function, \( f(x) \) is a known function, and \( K(x, t) \) is another known function of two variables \( x \) and \( t \).

*Erik Ivar Fredholm* (1866 -1927), Swedish mathematician, best known for his work in integral equations, spectral theory and equations of mathematical physics.

**Integration**  Finding a function by its derivative (or by its differential) is called integration. In other words, this is a process of finding \( f(x) \) by starting from \( f'(x) \). That means, integration is the operation inverse to differentiation. The correctness of integration can always be checked by differentiating the result.

For instance, \( \int 3x^2 \, dx = x^3 + C, \) since \( (x^3 + C)' = 3x^2. \)

**Integration by Change of Variable (by Substitution or The Reverse-Chain-Rule)** The integral of the form \( \int f(x)\, f'(x) \, dx \) can be often simplified if, instead of \( x \), we introduce a new variable of integration \( u \) by setting \( u = f(x) \).
Then \( du = f'(x)dx \) or \( dx = \frac{du}{f'(x)} \) and \( \int f(x)f'(x)dx = \int u \, du = \frac{u^2}{2} + C = \frac{f^2(x)}{2} + C, \) (1) where in the final result we returned to the ‘old’ variable \( x \). Formula (1) is called the formula for changing variable in indefinite integrals. Generally speaking, there are three forms of integrands where we can use integration by substitution \( u = f(x) \):

1. \( \int [f(x)]^n f'(x)dx = \int u^n \, du = \frac{u^{n+1}}{n+1} + C = \frac{f(x)^{n+1}}{n+1} + C, \; n \neq -1; \)
2. \( \int \frac{f'(x)}{f(x)} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |f(x)| + C; \)
3. \( \int e^{f(x)}f'(x) \, dx = \int e^u \, du = e^u + C = e^{f(x)} + C. \)

**Example 1** Find the integral \( \int x(4x^2 + 3)^5 \, dx \).

**Solution** If we choose \( u = 4x^2 + 3 \), then

\[
\frac{du}{dx} = 8x \rightarrow du = 8x \, dx \rightarrow dx = \frac{du}{8x} \rightarrow \int x(4x^2 + 3)^5 \, dx = \int xu^5 \, \frac{du}{8x} = \frac{1}{8} \int u^5 \, du = \frac{1}{8} \cdot \frac{u^6}{6} + C = \frac{u^6}{48} + C = \frac{(4x^2 + 3)^6}{48} + C.
\]

**Example 2** Find the integral \( \int \frac{x^3}{(x-1)^2} \, dx \).

**Solution** If we choose \( u = x - 1 \), then \( x = u + 1 \rightarrow dx = du \).

Now substitute.

\[
\int \frac{x^3}{(x-1)^2} \, dx = \int \frac{(u+1)^3}{u^2} \, du = \int \frac{u^3 + 3u^2 + 3u + 1}{u^2} \, du = \int u \, du + 3 \int \frac{du}{u} + \int \frac{du}{u^2} = \frac{u^2}{2} + 3u + \ln |u| - \frac{1}{u} + C.
\]

Since \( u = x - 1 \),

\[
\int \frac{x^3}{(x-1)^2} \, dx = \frac{1}{2} (x - 1)^2 + 3 (x - 1) + \ln |x - 1| - \frac{1}{x - 1} + C.
\]

**Example 3** Find the integral \( \int \frac{16x^3 + 3}{\sqrt{(4x^4 + 3x + 1)^2}} \, dx \).

**Solution** We set \( u = 4x^4 + 3x + 1 \). Then \( \frac{du}{dx} = 16x^3 + 3 \) or \( dx = \frac{du}{16x^3 + 3} \), and, consequently,

\[
\int \frac{16x^3 + 3}{\sqrt{(4x^4 + 3x + 1)^2}} \, dx = \int \frac{16x^3 + 3}{\sqrt{u^2}} \, \frac{du}{16x^3 + 3} = \int u^{-\frac{2}{3}} \, du = \frac{u^{-\frac{2}{3} + 1}}{-\frac{2}{3} + 1} + C = \frac{u^{\frac{1}{3}}}{\frac{1}{3}} + C = 3\frac{1}{3}u^{\frac{1}{3}} + C = 3\frac{1}{3}16x^3 + 3x + 1 + C
\]

**Integration by Parts** If \( u \) and \( v \) are differentiable functions, then \( \int u \, dv = uv - \int v \, du \). The process of finding integrals by the formula above is called integration by parts. This method/technique of integration can be used only if the integrand satisfies the following conditions.

1. The integrand can be written as the product of two factors, \( u \) and \( dv \).
2. It is possible to integrate \( dv \) to get \( v \) and to differentiate \( u \) to get \( du \).
3. The integral \( \int v \, du \) can be found.

This method often makes it possible to simplify/reduce a complicated integrand to a pretty simpler
The integrand, therefore, integral. If we have to repeat many times this method to find final answer, then we will use special “shortcut” – tabular or column integration. We will use tabular integration for integrals of the form $\int f(x) g(x) \, dx$, in which $f(x)$ can be differentiated repeatedly to become zero and $g(x)$ can be integrated without difficulty (See example in Accumulated Amount of Money Flow at Time).

**Example** Use integration by parts to find the integral $\int_1^9 \ln 3x \, dx$.

**Solution** Let $u = \ln 3x \rightarrow du = \frac{1}{3x} \cdot 3 \, dx = \frac{dx}{x}$; \, $dv = dx \rightarrow v = x$ \, $\Rightarrow$ \, $\int \ln 3x \, dx = uv - \int v \, du = x \ln 3x - \int x \cdot \frac{1}{x} \, dx = x \ln 3x - x$.

Therefore, $\int_1^9 \ln 3x \, dx = (x \ln 3x - x) \bigg|_1^9 = (9 \ln 27 - 9) - (1 \ln 3 - 1) = 9 \ln 3^3 - 9 - \ln 3 + 1 = 27 \ln 3 - 9 - 8 = 26 \ln 3 - 8 \approx 20.56$.

### Integration of Rational Function by Partial Fraction Decompositions

If the integrand is the rational expression (algebraic fraction) we will use the partial fraction decomposition to rewrite the integrand as a sum of simpler fractions (or partial fractions).

**Example** Evaluate the integral $\int_1 \frac{1}{x^2 - \sqrt{x}} \, dx$.

**Solution**

\[
\int_1 \frac{1}{x^2 - \sqrt{x}} \, dx = \int_1 \frac{1}{\sqrt{x}(x-1)} \, dx \quad \text{Let} \quad u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} \, dx \Rightarrow dx = 2\sqrt{x} \, du = 2udu
\]

\[
= \int_1 \frac{2u}{u(u^2 - 1)} \, du = \int_1 \frac{2}{u(u+1)(u-1)} \, du;
\]

To evaluate this integral we have to find the form of the partial fraction decomposition of the integrand

\[
\frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow 2 = A(u-1) + B(u+1);
\]

First, we will substitute $-1$ for $u$ to get $A$ : \, $2 = A(-1 - 1) + B(-1 + 1) \Rightarrow A = -1$.

Replace $A$ with $-1$ and substitute $1$ for $u$ to get $B$.

$2 = -1(1 - 1) + B(1 + 1) \Rightarrow B = 1$.

Therefore, \[\int_1 \frac{2}{(u+1)(u-1)} \, du = \int_1 \frac{-1}{u+1} \, du + \int_1 \frac{1}{u-1} \, du = -\ln |u + 1| + \ln |u - 1| + C = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \right| + C.\]

### Integral Test

If a function $f$ is positive, continuous, and decreasing for $x \geq 1$, and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) \, dx$ either both converge or both diverge.

**interest** A percentage of an amount invested or borrowed. In other words, this is a charge for the use of money, paid by the borrower to the lender. Interest can be thought of as “rent of money”, defined as the compensation paid by the borrower of money to the lender of money.

**interest rate** The percentage of the principal that is paid as a fee over a certain period of time (one month or year).
Intermediate Value Theorem If \( f(x) \) is a polynomial function with only real coefficients, and if for real numbers \( a \) and \( b \) the values \( f(a) \) and \( f(b) \) are opposite in sign, then there exists at least one real zero between \( a \) and \( b \).

interpolation The process of estimating a value between given values.

intersections of sets A and B Intersection of two sets A and B is the set of all elements that are common to both sets A and B. It is denoted by \( A \cap B \) (is read ‘A intersection B’) or, using set-builder notation, \( A \cap B = \{x | x \in A \text{ and } x \in B\} \).

interval An interval is a portion of the real number line, which may or not include its endpoint(s).

interval notation The use of a pair of real numbers inside parentheses and brackets to represent the set/interval of real numbers between those two numbers (see also closed and open intervals).

invariant Properties of a figure that stay the same regardless of how the figure is deformed.

inverse Opposite or reverse.

inverse of conditional The statement formed by negating both the hypothesis an conclusion of a conditional statement.

Inverse function Let \( f \) be a one-to one function. Then \( g \) is the inverse function of \( f \) if
\[
(f \circ g)(x) = x \quad \text{for every } x \text{ in the domain of } g, \quad (g \circ f)(x) = x \quad \text{for every } x \text{ in the domain of } f.
\]
If \( g \) is the inverse of function \( f \), then \( g \) is written as \( f^{-1} \) (read “\( f \) – inverse”, and do not confuse the \(-1\) in \( f^{-1} \) with a negative exponent).

Briefly speaking, the inverse function of the one-to-one function \( f \) is defined as
\[
f^{-1} = \{(y, x) | (x, y) \in f\}.
\]

N. B. The domain of \( f \) is the range of its inverse function \( f^{-1} \), and the range of \( f \) is the domain of its inverse function \( f^{-1} \). That means if the point \((a, b)\) lies on the graph of \( f \), then the point \((b, a)\) lies on the graph of its inverse function \( f^{-1} \). In other words, if we interchange the argument \((x)\) and the function \((f)\) in a given functional relation, we obtain* a new function \((f^{-1})\), the inverse of the original function.

* Finding the equation of the inverse of \( y = f(x) \)
For one-to-one function \( f \) defined by an equation \( y = f(x) \), find the defining equation of the inverse using following steps . First of all, you will replace \( f(x) \) with \( y \) if you have that case, and after that:

\begin{itemize}
  \item \textbf{Step 1} Interchange \( x \) and \( y \);
  \item \textbf{Step 2} Solve equation for \( y \) (isolate \( y \));
  \item \textbf{Step 3} Replace \( y \) with \( f^{-1}(x) \).
\end{itemize}

Example Decide whether equation defines a one-to-one function. If so, find the equation of the inverse. \( f(x) = 3x + 1 \)

\textbf{Solution} The graph of \( y = 3x + 1 \) is a non-horizontal straight line, so by horizontal line test, we can see
that the given function is a one-to-one function. To find the equation of the inverse we will replace \( f(x) \) with \( y \).

\[
y = 3x + 1
\]

\[
x = 3y + 1
\]  \((\text{Step 1: Interchange } x \text{ and } y)\)

\[
3y = x - 1
\]  \((\text{Step 2: Solve for } y)\)

\[
y = \frac{x - 1}{3}
\]

\[
f^{-1}(x) = \frac{x - 1}{3} \quad (\text{Step 3: Replace } y \text{ with } f^{-1}(x))
\]

**Inverse operations** Two operations that cancel (“undo”) each other. For example, addition/subtraction are inverse operations or multiplication/division or raising to a power/taking the root or differentiation/integration or ...

**Inverse relation** The relation formed by interchanging the members of the domain and the range of a given(initial) relation.

**Inverse variation** A rational function defined by an equation of the form \( y = \frac{k}{x} \), where nonzero number \( k \) is so called the constant of variation. N.B. In inverse variation, an assignment of increasing absolute value for \( x \) produces decreasing absolute value for \( y \). In other words, we will say that \( y \) varies inversely as \( x \), or that \( y \) is inversely proportional to \( x \).

**Invalid argument** An argument that is not valid is invalid, or fallacy.

**Involution** Raising a number to a (integral) power.

**Irrational equations/radical equations** Any equation where the variable/unknown value is inside a radical is called an irrational/radical equation.

**Example** Solve equation.

\[
\sqrt{x + 1} - x = 1
\]

**Solution**

1st step: We have to isolate the radical.

\[
\sqrt{x + 1} = x + 1
\]

2nd step: We have to square (or rise to a power) both sides of the equation to remove the radical.

\[
(\sqrt{x + 1})^2 = (x + 1)^2 \quad \rightarrow \quad x + 1 = x^2 + 2x + 1 \quad \rightarrow \quad x^2 + x = 0 \quad \rightarrow \quad \text{This quadratic equation we can solve by factoring (easiest way) } \rightarrow \quad x(x + 1) = 0 \quad \rightarrow \quad x_1 = 0, \ x_2 = -1
\]

3rd step: We have to verify proposed solutions.

\[
x_1 = 0: \quad \sqrt{0 + 1} - 0 = 1 \quad \rightarrow \quad 1 = 1 \quad (\text{True statement}) \quad x_2 = -1 \quad \rightarrow \quad \sqrt{-1 + 1} - (-1) = 1 \quad \rightarrow \quad 1 = 1 \quad (\text{True statement})
\]

Therefore, the solution set is \{-1, 0\}. N.B. If the irrational equation has several radicals, we have to repeat the first two steps of the process to remove all of them.

**Irrational number/surd** A real number that cannot be expressed as a ratio (fraction) of two integers. The term “irrational” literally means “having no ratio”. For example, \( \sqrt{2}, \sqrt{5}, -\sqrt{3}, \emptyset, e, \pi \) are irrational.
numbers**.

*Famous Persian mathematician **Mohamed ibn-Musa al Khwarizmi** (around 820 AD) called irrational numbers “inaudible” ... this was later translated to the Latin *surdus* (“deaf” or “mute”), and more later to the English *surd*. Al Khwarizmi referred to the rational numbers as “audible”. He made major contributions in the fields of algebra, trigonometry, astronomy, astrology, geography and cartography, and this outstanding Persian mathematician was one of the first Directors of the *House of Wisdom* in Bagdad in the early 9th Century.

**In 1874 the famous German mathematician **Georg Cantor** (1845-1918) made the startling discovery that there are more irrational numbers than rational ones, and more transcendental numbers than algebraic ones.

**isolated points** In topology, a branch of mathematics, a point $x$ of a set $A$ (or $x \in A$) is called an isolated point of $A$ if there exists a neighborhood of $x$ not containing other points of $A$.

**isosceles triangle** A triangle that has at least 2 congruent sides.

**Isosceles right triangle** A right triangle in which both legs have the same length.

**iteration** The repetitive application of the same rule.

**J**

**jerk/jolt** A sudden change in acceleration is called *jerk*. In other words, the jerk $j$ is the rate of change of acceleration, or the first derivative of acceleration or the second derivative of the velocity. The jerk actually is the third derivative of position with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}.$$  

**joint variation** An equation of the form $y = kxwz$, where $k$ is a nonzero constant, so called *constant of variation*. We will say that $y$ varies jointly as $x$, $w$, and $z$. In other words, *joint variation* is the case of *direct variation* when a variable ($y$) varies directly with more than one another variable ($x$, $w$, $z$). For example, the volume of a right circular cylinder, i.e. $V = \pi r^2 h$, varies jointly as $h$ and the square of $r$.

**K**

**Kepler’s Laws** The German astronomer **Johannes Kepler** (1571 – 1630) formulated three laws that describe the motion of planets about the sun in our solar system.

**First Law.** The orbit/path of each planet is an ellipse with the sun at one focus.

**Second Law.** The vector from the sun to the planet sweeps out equal areas in equal times or the vector
from the sun to a moving planet sweeps out area at a constant rate.

**Third Law.** If the time required for a planet to travel once around its elliptical orbit is \( T \) (orbital period), and if the major axis of the ellipse is \( 2a \), then

\[ T^2 = k a^3 \]

for some constant \( k^* \). In other words, the square of the orbital period is proportional to the cube of the mean distance between the planet and the sun.

\[ k = \frac{4\pi^2}{GM} \]

where \( G \) is universal gravitational constant (about \( 6.6726 \cdot 10^{-11} \text{ Nm}^{-2} \text{kg}^{-2} \)), and \( M \) is sun’s mass (\( 1.99 \cdot 10^{30} \text{ kg} \)).

**kite** A quadrilateral with exactly two pairs of adjacent congruent sides. Note that the diagonals of a kite are perpendicular.

The area of a kite is given by

\[ A = \frac{1}{2} d_1 d_2 \]

where \( d_1 \) is a long diagonal of kite, and \( d_2 \) is a short diagonal of kite.

**Kolmogorov’s Axioms** The Kolmogorov* axioms state that the probability of any event is equal or greater than zero, and the probability certain event is 1. If \( E \) and \( F \) are two mutually exclusive events (events that cannot both occur), then the probability of disjunction (the probability of either \( E \) or \( F \) occurring) is equal to the sum of their probabilities, i.e., \( P(E \text{ or } F) = P(E) + P(F) \).

…………………………

*Andrey Kolmogorov (1903-1987), famous Russian mathematician who made significant contribution especially to the mathematics of probability theory and topology.

**kth remainder of series** See *number series*.

**L**

**Lagrange’s Mean Value Theorem** Let \( f: [a, b) \rightarrow R \) be a continuous function, differentiable on the open interval \((a, b)\). Then there is some \( c \in (a, b) \) such that

\[ f'(c) = \frac{f(b)-f(a)}{b-a} \]

This is one of the most important results in mathematical analysis/calculus says that differentiable function must at some point grow with instantaneous velocity equal to its average velocity over an interval. Integral version of this theorem is so called Mean Value Theorem.

……………………

*Joseph-Louis Lagrange (1736 – 1813), Italian mathematician and astronomer.

**Laplace transform** Given a function \( f(t) \) defined for all \( t \geq 0 \), the Laplace transform of the function \( f(t) \) is the function \( F(s) \) defined as follows:
\( F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt \)

for all values of \( s \) for which the given improper integral converges.

\textit{N.B.} New function \( F(s) \) is a function of a new independent variable \( s \). The Laplace can be used to solve initial value problems for linear differential equations with constant coefficients, especially if we can’t use any other method. The Laplace transform is named for the French mathematician \textbf{Pierre-Simon de Laplace} (1749 – 1827), who studied this transform in 1782.

**latitude** Angular distance north or south from the Earth’s equator measured through 90°. In other words, latitude gives the measure of a central angle with vertex Earth’s center whose initial side goes through the equator and whose terminal side goes through the given location.

**Example** Find the distance in kilometers between Panama City, Panama, 9° N, and Pittsburg, USA, 40° N, assuming they lie on the same north-south line.

**Solution** The central angle between Panama City and Pittsburg is \( \theta = 40° - 9° = 31° \). The distance between the two cities can be found by formula for arc length of a circle \( s = r\theta \), after 31° is first converted to radians.

\[ 31° = 31\left(\frac{\pi}{180}\right) \text{ radian} = \frac{31\pi}{180} \text{ radian} \]

\[ s = r\theta = 6400\left(\frac{31\pi}{180}\right) \approx 3500 \text{ km} \]  \textbf{(Note.} The radius of Earth is 6400 km) \textbf{)}

**law of cosines** In any triangle ABC, with sides \( a, b, \) and \( c, \)

\[ a^2 = b^2 + c^2 - 2bc \cos A, \]

\[ b^2 = a^2 + c^2 - 2ac \cos B, \] \text{ and }

\[ c^2 = a^2 + b^2 - 2ab \cos C. \]

**law of sines** In any triangle ABC, with sides \( a, b, \) and \( c, \)

\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \] \text{ where } \( R \) denotes the radius of a circumscribed circle.

\textbf{N. B.} \textit{The smallest angle is opposite the shortest side, and the largest angle is opposite the longest side} (assuming the triangle has sides that are all of different lengths).

**leading coefficient** The coefficient of the term of highest degree in a polynomial.

**leading term** The term of highest degree in a polynomial.

**least common denominator (LCD)** The least common multiple of the denominators of the given fractions is called the \textit{least common denominator} (LCD). The LCD of two or more fractions (rational expressions) is found as follows:

1. Factor each denominator completely.
2. Identify each different prime factor from all the denominators.
3. Form the product using each different factor to the highest power that occurs in anyone denominators.

**Example** Combine into a single fraction and reduce to lowest terms.
Solution To find LCD, factor each denominator completely (prime factorization):

\[ 10 = 2 \cdot 5; \quad 6 = 2 \cdot 3; \quad 45 = 3^2 \cdot 5 \Rightarrow \text{LCD (10, 6, 45)} = 2 \cdot 3^2 \cdot 5 = 90. \]

Now use the fundamental property of fractions to make each denominator 90:

\[
\frac{7}{10} + \frac{5}{6} - \frac{13}{45} = \frac{9 \cdot 7 - 15 \cdot 5 - 2 \cdot 13}{90} = \frac{63 + 75 - 26}{90} = \frac{112}{90} = \frac{56}{45}.
\]

N. B. The ladder method (with factor tree)

2 | 6, 10, 45
3 | 3, 5, 45
3 | 1, 5, 15
5 | 1, 5, 5
\[ \Rightarrow 1, 1, 1 \]

\[
\text{LCD (10, 6, 45)} = 2 \cdot 3^2 \cdot 5 = 90.
\]

Least common multiple (LCM) The smallest number that is multiple of the given numbers. For example, \( \text{LCM (4, 6)} = 12. \)

N. B. LCM (4, 6) = LCD (4, 6) = 12, or generally, \( \text{LCM (a, b)} = \text{LCD (a, b)}, \forall a, b \in \mathbb{Z}. \)

Least Squares The method of least squares is a standard approach to the approximate solution of overdetermined systems, i.e. sets of equations in which there are more equations than unknowns (See below Least Squares Line).

Least Squares Line A straight line \( y = mx + b \) that gives best fit to the data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) has slope \( m \) and y-intercept \( b \) given by

\[
m = \frac{n(\sum xy) - (\sum x)(\sum y)}{n(\sum x^2) - (\sum x)^2} \quad \text{and} \quad b = \frac{\sum y - m(\sum x)}{n}.
\]

Leibniz’s Principle of Sufficient Reason (PSR) For every proposition \( p \), if \( p \) is true, then there is a sufficient explanation why \( p \) is true. The Principle of Sufficient Reason in its classic form is simply that nothing is without a reason (Nihil Est Sine Ratione) or, in other words, there is no effect without a cause. Leibniz once said that without the principle of sufficient reason, very little in philosophy and science could be demonstrated.

Length The length of a segment is the measure of the distance from one endpoint to the other.

Level curve The set of points in the plane where a function \( f(x, y) \) has a constant value \( f(x, y) = c \) is called a level curve of \( f \).

Level surface The set of points \((x, y, z)\) in space where a function \( f(x, y, z) \) has constant value \( f(x, y, z) = c \) is called a level surface of \( f \).
light year or lightyear (ly) Light year is a unit of distance. It is the distance that light can travel in one year with a velocity of about 300,000 km per second. In other words, 1 ly = \(9.4605284 \times 10^{12}\) km = \(5.88 \times 10^{12}\) miles.

like radicals Radicals/roots that have the same indices and the same radicands.

like terms The terms in which the variable factors are the same. For example, in expression \(7x^3 - 5x + 3x^2 + 9x - 11\) the terms \(9x\) and \(-5x\) are like terms because the variable part of each term is the same. In other words, like terms are terms whose variables and exponents match.

Limit of a Function at a Point The number or point \(L\) that is approached by a function \(f(x)\) as \(x\) approaches \(a\), i.e. \(\lim_{x \to a} f(x) = L\).

N.B. A function cannot have two different limits at one point.

* Technique for calculating \(\lim_{x \to a} f(x)\)

1) Try substituting \(a\) into the limit expression. If you can evaluate/solve this expression you’re done. This procedure/method only works when the function is continuous at \(x = a\).

2) If after you substitute, you can’t simplify, try simplifying algebraically first (factoring, rationalizing the numerator/denominator, ...) then substitute.

3) If you get a non-zero number for the numerator and a zero in the denominator, then there is no limit.

4) If you get zero in the numerator and denominator keep working, there is limit.

5) To evaluate limit of a rational function at infinity \((x \to \infty)\) divide numerator and denominator by the highest power of \(x\) that shows up. Then substitute \(\infty\) for \(x\) to find the limit and use the rules for limits at infinity.

\[
\begin{align*}
\text{a)} & \quad \lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{b)} & \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0
\end{align*}
\]

6) L’HOSPITAL’S RULE*  
Let two functions \(f(x)\) and \(g(x)\) be differentiable on the interval \((a, b)\) and vanish at a point \(x = c, \ c \in (a, b)\). Then
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{f''(x)}{g''(x)} = \ldots, \text{ provided that the last limit exists.} 
\]

* Evaluating limits by this formula is called L’Hospital’s [Guillaume Francois Antoine Marquis de l’Hospital (1661-1704), French mathematician who wrote the first textbook on calculus] rule for evaluating indeterminate forms \(\frac{0}{0}\) and \(\frac{\infty}{\infty}\). This rule we can use for evaluating other indeterminate forms \((0 \cdot \infty, \ 0^0, \ 1^\infty, \ \infty^0, \ \infty - \infty)\) by converting them to the principal types \(\frac{0}{0}\) or \(\frac{\infty}{\infty}\) by identity transformations. For \(0^0, \ \infty^0, \ \text{and} \ 1^\infty\) we will use two steps:

1** step: We will reduce each of the indeterminate forms to an indeterminate form of type \(0 \cdot \infty, \ \frac{0}{0}, \ \text{or} \ \frac{\infty}{\infty}\).
2nd step: We will reduce the indeterminate form \(0 \cdot \infty\) to an indeterminate form of type \(\frac{\infty}{\infty}\) or \(\frac{0}{0}\).

Keep always in mind: \(\frac{a}{0} = \infty\) and \(\frac{a}{\infty} = 0\), \(a \neq 0\)

Commonly Occurring Limits

1) \(\lim_{x \to 0} \frac{\sin x}{x} = 1\)  
2) \(\lim_{x \to 0} \frac{\sin ax}{ax} = 1\)  
3) \(\lim_{x \to 0} \frac{\sin^n ax}{(ax)^n} = 1\)  
4) \(\lim_{x \to \infty} \frac{\ln x}{x} = 0\)  
5) \(\lim_{x \to \infty} \frac{n}{\sqrt{n}} = 1\)  
6) \(\lim_{x \to \infty} x^n = 0\), \(|x| < 1\)  
7) \(\lim_{x \to \infty} \frac{x^n}{n!} = 0\)  
8) \(\lim_{n \to \infty} x^n = 1\), \(x > 0\)  
9) \(\lim_{n \to \infty} (1 + \frac{1}{n})^n = e\)  
10) \(\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x\)

Line graph A graph in which quantities are represented as points connected by straight-line segments.

Line integral The line integral of \(f(x, y, z)\) over the curve \(C\) from \(a\) to \(b\) is the definite integral of the function \(f(g(t), h(t), k(t))\) with respect to the variable \(t\) between \(a\) and \(b\). The line integral is denoted by the symbol

\[
\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t), k(t))|v(t)|dt, \text{ where } v(t) = \frac{dr}{dt}, \text{ and } r(t) = g(t)i + h(t)j + k(t)k
\]

Line segment A set of points containing two points and all the points between them in a straight line. In other words, the line segment is a part of a straight line defined by two endpoints.

Line/axis of symmetry A straight line that divides a figure into two congruent halves that are mirror images of each other.

Linear cost function A linear function that has the form \(C(x) = mx + b\), where \(x\) represents the number of items produced, \(m\) represents the variable cost per item, and \(b\) represents the fixed cost.

Linear equation (first-degree equation) in one variable An equation that can be written in the form \(ax + b = 0\), where \(a\) and \(b\) are real numbers with \(a \neq 0\). Generally speaking, a first-degree equation in any number of unknowns is also called a linear equation.

Linear function A function whose graph is a straight line and it is defined by the linear (1st degree) equation of the form \(Ax + By = C\) (standard form), \(B \neq 0\) or \(y = mx + b\) (explicit form or slope-intercept form).

N. B. Linear functions are described by a linear equations and these equations are often written in the standard form \(Ax + By = C\). But, if \(B = 0\) we don’t have a function, because for only one x-value \((x = \frac{C}{A})\) we will have infinite number of y-values (vertical line).

Linear regression A method of fitting a straight line to a set of data.
linear speed $v$ The linear speed $v$ measures the distance $d$ traveled per unit of time $t$ and is defined by
\[ v = \frac{d}{t}. \]

linear systems of equations Two or more linear (1st degree) equations that contains the same variables (unknowns).

linearization If $y = f(x)$ is differentiable function at $x = a$, then approximating function $L(x) = f(a) + f'(a)(x - a)$ is the linearization of $y = f(x)$ at $x = a$. The approximation $f(x) \approx L(x)$ of $y = f(x)$ by $L(x)$ is the standard linear approximation of $y = f(x)$ at $x = a$. The point $x = a$ is the center of the approximation.

List of Symbols

$N$ The set of natural numbers, i.e. $N = \{1, 2, 3, \ldots\}$.

$W$ The set of whole numbers, i.e. $W = \{0, 1, 2, 3, \ldots\}$.

$Z$ The set of integers, i.e. $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$Q$ The set of rational numbers (fractions).

$I$ The set of irrational numbers.

$R$ The set of real numbers.

$C$ The set of complex numbers.

$\emptyset$ An empty set.

$\infty$ John Wallis (1616 – 1703), English mathematician, is credited with introducing the present symbol $\infty$ for infinity.

$\text{APR}$ An annual percentage rate.

$a \in M$ The element/number $a$ belongs to the set $M$ or $a$ is element of $M$.

$a \notin M$ The element/number $a$ does not belong to the set $M$ or $a$ is not element of $M$.

$\{a, b, c, d\}$ A set consisting of the elements $a, b, c, d$.

$\exists x$ There is an $x$ (Existential quantifier).

$\exists! x$ There is only one $x$.

$\therefore$ ... therefore ... \((x = y \text{ and } y = z; \therefore x = z)\)
∀x  For any x (Universal quantifier).

A ⇒ B  B follows from A (Logical implication symbol).

A ⇔ B  A if and only if B or B follows from A and, conversely, A follows from B. (Logical equivalence symbol).

A ∪ B  The union of the sets A and B.

A ⊆ B  A is subset of B (subset symbol).

⊂  Proper subset symbol.

A ∩ B  The intersection of the sets A and B.

A \ B  The difference of the sets A and B.

a = b  The equality sign (“a is equal to b”).

≈  Approximate equal sign.

a ≠ b  The comparison sign, i.e. a is not equal to b.

<, >, ≤, ≥  The inequality signs (is less than, is greater than, is less than or equal, is greater than or equal).

≅  Congruency sign (ΔABC ≅ ΔDEF).  G. W. Leibniz (1646 – 1716) developed symbol for congruence. He also created first the mechanical calculator capable of multiplication and division.

~  Similarity sign (ΔABC ~ ΔDEF).

a ∝ b  The symbol for proportionality (“a is proportional to b”).

[a, b]  A closed interval beginning at a and ending at b.

(a, b)  An open interval beginning at a and ending at b.

|a|  The absolute value/magnitude of the number a.

√x  The square root of x.

√x  The nth root of x.

n!  The product of the first n natural numbers, i.e.  n! = 1·2·3···n.

∑}_{i=1}^{n} a_i  The summation notation, i.e.  ∑}_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n.
\[\lim_{x \to a} f(x) = b\] The number \(b\) is a limit of the function \(f(x)\) as \(x\) approaches/tends to \(a\).

\[x\] The integral part of the number \(x\)

\(\Delta y\) The increment of a function \(y\) at a point (Indicates a small change).

\(\triangle ABC\) Triangle symbol (Denotes vertices of triangle).

\(dy\) The differential of the function \(y(x)\).

\(f'(x)\) or \(y'(x)\) The derivative of the function \(y = f(x)\) at the point \(x\).

\(f^{(n)}(x)\) The \(n\)th derivative of the function \(f(x)\).

\[\int f(x)\,dx\] The indefinite integral of the function \(f(x)\).

\[\int_a^b f(x)\,dx\] The definite integral of the function \(f(x)\) over the closed interval \([a, b]\).

\(N.B.\) Nota bene (Lat): Mark well.

Q.E.D. Quod erat demonstrandum (Lat): Which was to be proved. Q.E.D. is used in mathematical proofs to show that what was to be proven has been proven.

***************

*In Robert Recorde's book The Whetstone of Witte, published in 1557, we find the first use of the equals sign, \(=\) ('to avoid the tedious repetition', he said, the phrase - 'is equal to').

** Thomas Harriot (Oxford, 1560 – London, 1621) was first well known mathematician to set foot in Columbus's India 1585. He used first the symbols for inequality \(<\) (“less than”), and \(>\) (“greater than”). He wrote square of number \(A\) as \(AA\), cube of number \(A\) as \(AAA\), \(\text{and 4th power of A as AAAA. That was literally first great step. Herriot was the first person to make a drawing of the Moon through telescope, on July 28, 1609, over four months before Galileo Galilei (1564-1642). He is also credited with the introduction of the potato (!) to Great Britain and Ireland.}

locus* When a point moves so as to comply with certain conditions, the path it traces out is called the locus of the point under these condition.

***************

*Pierre de Fermat (1601-1665):“Whenever two unknown magnitudes appear in a final equation, we have a locus, the extremity of one of the unknown magnitudes describing a straight line or curve.”

logarithm* The logarithm of a number \(N\) to a base \(a\) is the exponent \(x\) indicating the power to which \(a\) must be raised to obtain \(N\), i.e. \(\log_a N = x\). Symbolically, \(\log_a N = x\) is equivalent to \(a^x = N\) or \(a^{\log_a x} = x\) (Basic logarithmic identity)

For example, \(\log_2 16 = 4\) because \(2^4 = 16\) (or \(5^{\log_5 7} = 7\)).

Other properties of logarithms
a) \( \log_a xy = \log_a x + \log_a y \) because \( a^x \cdot a^y = a^{x+y} \);
b) \( \log_a x^\frac{y}{x} = \log_a x \cdot \log_a y \) because \( a^x \cdot a^y = a^{x-y} \);
c) \( \log_a x^n = n \log_a x \) because \( \log_a (x \cdot x \cdot \cdots x) = \log_a x + \log_a x + \cdots + \log_a x = \cdots (n \text{ addends/summands}) \cdots = n \log_a x \);
d) \( \log_a \sqrt[n]{x^m} = \frac{m}{n} \log_a x \) because \( \sqrt[n]{a^m} = a^{\frac{m}{n}} \);
e) \( \log_a a = 1 \) because \( a^1 = a \) & \( \log_a 1 = 0 \) because \( a^0 = 1 \).

**The Change-of-Base Formula** \( \log_a M = \frac{\log_b M}{\log_b a} \)

**N. B.** Base \( a \) and number \( N \) (so called logarithmand or argument) are positive numbers, \( a \neq 1 \).

*The invention of logarithms is credited to the Scotsman John Napier (1550-1617), who first called logarithms “artificial numbers”. Later he joined the Greek words logos (ratio) and arithmos (number) to form the word used today. The development of logarithms was motivated by a insistent need for faster computation, especially if we deal with so small or so large numbers.*

**logarithmic equation** An equation containing a logarithmic expression.

**Example 1** Solve equation \( \log x + \log (x - 21) = 2 \).

**Solution** To solve this logarithmic equation we need on the both sides the single logarithm with coefficient 1 (using properties of logarithms) and check the proposed solution. Therefore, \( \log x(x - 21) = \log 100 \) (We used here product property and fact that \( 2 = \log 100 \)) →

\[
x(x - 21) = 100 \quad \Rightarrow \quad x^2 - 21x - 100 = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{21 \pm \sqrt{441 + 400}}{2} = \frac{21 \pm 29}{2} \quad \Rightarrow \quad x_1 = 25, \quad x_2 = -4
\]

Since the negative solution \( x = -4 \) is not in the domain of \( \log x \) (logarithmand \( x \) must be positive number), it must be discarded. Therefore, the only valid solution is the positive number 25, giving the solution set is \{25\}.

**Example 2** Solve equation \( \log^2 x - 3 \log x + 2 = 0 \).

**Solution** This type of logarithmic equation is quadratic in form, and with the substitution \( u = \log x \) we will have \( u^2 - 3u + 2 = 0 \) → \( (u - 2)(u - 1) = 0 \) → \( u - 2 = 0 \) or \( u - 1 = 0 \) (zero-factor property) → \( u_1 = 2, \quad u_2 = 1 \). Just now we will replace \( u \) with \( \log x \). Therefore, \( \log x = 2 \) → \( x = 10^2 \) (definition of logarithm) → \( x_1 = 100 \) or \( \log x = 1 \) → \( x_2 = 10 \).

Check: \( \log^2 x - 3 \log x + 2 = 0 \) (Original equation)

(a) If \( x = 100 \), then

\[
(\log 100)^2 - 3 \log 100 + 2 = 0 \quad (?)
\]

(b) If \( x = 10 \), then

\[
(\log 10)^2 - 3 \log 10 + 2 + 0 \quad (?)
\]

\[
2^2 - 3 \cdot 2 + 2 = 0 \quad (?)
\]

\[
1 - 3 + 2 = 0
\]

\[
4 - 6 + 2 = 0 \quad \Rightarrow \quad 0 = 0 \quad \text{(True)}
\]

Both check, so the solution set is \{10, 100\}.

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**logarithmic function** If $a > 0$, $a \neq 1$, and $x > 0$, then $f(x) = \log_a x$ defines the logarithmic function with base $a$. It is the inverse of an exponential function $f(x) = a^x$.

**logic** The science which directs the operations of the mind in the attainment of truth.

**logic gate** An electronic circuit that represents *not*, *or*, and *and*.

**logical chain** A series of logically linked conditional statements.

**logically equivalent statements** Two statements, each of which can be logically derived from the other.

**logical thinking** Reasoning in accordance with principles*/rules of logic.

*Principles of logic: 1) **Existence** (Everything is something.); 2) **Identity** (A thing is the thing it is.); 3) **Uniqueness** (No thing is another thing than the thing it is.); 4) **Specificity** (Everything has some property); 5) **Excluded middle** (A thing has or does not have a particular property.); 6) **Non-contradiction** (No thing has and does not have a particular property.).

**longitude** The angular distance (expressed in degrees) East or West of the prime meridian, which goes from the South Pole to the North Pole through Greenwich, England.

**lower quartile** The median of the lower half of an ordered set of data.

**lowest terms** An fraction is in the lowest terms (or in the simplest form) if its numerator and denominator have no factors in common.

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M

**Maclaurin Series** See *Taylor Series*.

**magnitude/length of a vector** The numerical measure of a vector. The magnitude (length/absolute value) of a vector $u = \langle a, b \rangle$ is given by the formula $|u| = \sqrt{a^2 + b^2}$.

**major axis of an ellipse** The longer axis of symmetry.

**mantissa** The decimal part of a logarithm. For example. Common logarithm of 159 is 2.201397124 or log 159 = 2.201397124. The mantissa is 0.201397124, while the integral part 2 is known as characteristic of the logarithm.
**marginal average cost** If the total cost to manufacture \( x \) items is given by function \( C(x) \), then average cost per item is \( \bar{C}(x) = \frac{C(x)}{x} \). The **marginal average cost** is the derivative of the average cost function, \( \bar{C}'(x) \).

**marginal cost** In economical sciences, marginal cost is the rate of change of the cost function \( C(x) \) at a level of production \( x \) items. If the cost function is the linear function of the slope-intercept form \( C(x) = mx + b \), the slope \( m \) represents the **marginal cost** per item. Generally speaking, the marginal cost is the derivative of \( C(x) \), i.e. \( C'(x) \).

**marginal revenue** Marginal revenue refers to the rate of change of revenue function \( R(q) \). Since the derivative of a function gives the rate of change of the function, a marginal revenue (or cost or profit) function is found by taking the derivative of the revenue (or cost or profit) function. In other words, the marginal revenue is \( R'(q) \).

**marked price** The original price of an item.

**Marko’s Rule:** “We can’t add frogs and dogs.” In other words, we can combine only like terms.

**Markov chain** A system for which the probability of transition to the next state depends only on its present state. In other words, it does not depend on its past history, except in so far as that history is coded in its present state.

**markup** The markup on a consumer item is the difference between the cost a retailer pays for an item and the price at which the retailer sells the item.

**Mathematical Analysis** Mathematical analysis, in the broadest sense of the term, includes very large part of the mathematics. It includes differential and integral calculus, the theory of functions of real and complex variable, the theory of ordinary and partial differential equations, the theory of integral equations, and certain other mathematical disciplines.

**mathematical induction** A method for proving that certain statements or formulas \( S_n \) are true for all natural numbers /positive integers.

**mathematical model** A mathematical description of the real-life situation. Before using mathematics to real-life problem, we must set up a **mathematical model**. In other words, a mathematical model is a translation of a real-world or word problem to a symbolic form suitable for mathematical analysis. To translate the verbal description of the problem means to get an equation or system of the two or more equations. This process of the finding equation(s) is called **mathematical modeling**. The equation (sometimes inequality) itself is often called a **mathematical model** of the real situation described in the word problem.

**Mathematical Physics** The application of mathematics to problems in physics.
The relationship between mathematics and physics has a long history. Traditionally, mathematics provides the language physicists use to describe nature, while physics brings mathematics to life: To discover the laws of mechanics, Newton needed to develop calculus. The very idea of a precise physical law, and mathematics to describe it, were born together. To unify gravity and special relativity, Einstein needed the language of Riemannian geometry. He used known mathematics to discover new physics. General relativity has been inspiring developments in differential geometry ever since. Quantum physics impacted many branches of mathematics, form geometry and topology to representation theory and analysis, extending the pattern of beautiful and deep interactions between physics and mathematics throughout centuries. [Mina Aganagic, STRING THEORY AND MATH: WHY THIS MARRIAGE MAY LAST. MATHEMATICS AND DUALITIES OF QUANTUM PHYSICS, Bulletin of the American Mathematical Society, January 2016]

Mathematics  Mathematics* is counting of definite things (Carl Friedrich Gauss:” Mathematics is the queen of the sciences and number theory is the queen of mathematics.”). In other words, mathematics** is the systematic treatment of magnitudes, relationships between figures and forms, and relations between quantities expressed symbolically. The word mathematics comes from Greek word “mathema”, which means “lesson”.

“Mathematics is very close to Navajo culture: both are deeply rooted in the love of beauty” (Henry Fowler, the son of a Navajo code talker, a mathematician and Navajo scholar from Dine College, Notices of the American Mathematical Society, August 2016).

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*”Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.” [Bertrand Russell (1872-1970), Mysticism and Logic, 1917]  
** 1. Galileo Galilei (1564-1642):”Mathematics is the language in which God has written the universe.”  
2. Nikolay Lobachevsky (1792 – 1856):“There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.”  
4. David Hilbert (1862-1943): “Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.”  
5. James Joseph Sylvester (1814-1897): “Mathematics is the music of reason”.  
6. Carl Friedrich Gauss (1777- 1855):“In mathematics there are no true controversies.”  
7. Simeon Denis Poisson (1781 - 1840):”Life is good for only two things, discovering mathematics and teaching mathematics”.

mathematics education*  The complex and heterogeneous social system, according Juan Godino and Carmen Batanero (1997), which includes theory, development, and practice concerning the teaching and learning** of mathematics.
*1. Jean Jacques Rousseau (1712 – 1778): “Plants are fashioned by cultivation, man by education.”
2. Benjamin Franklin (1706 - 1790): “Tell me and I forget. Teach me and I remember. Involve me and I learn.”
** George Polya (1887 - 1985): “Learning begins with action and perception, proceeds hence to words and concepts, and should end in good mental habits.”

matrix A matrix is a rectangular array or scheme of numbers, symbols or expressions, arranged in rows and columns. For example, the dimensions of the matrix below are $2 \times 3$ (read “two by three”) because there are two rows and three columns.

\[
\begin{bmatrix}
3 & 5 & 7 \\
6 & 2 & 9
\end{bmatrix}
\]

maximum value The largest function value (output) achieved by a function.

Mayan Numerals The Mayans developed a sophisticated numeration system with the base 20 (or vigesimal number system). The numerals in this place-value system are written vertically, and the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 of the Mayan numeration system are formed by a simple grouping dots and horizontal bars (line segments). The Mayan numerals was in use by 1000 BC.
**mean** The sum of a set of numbers/quantities divided by the number of addends (also called *average*).

**mean quantities** If we have a sequence of quantities (numbers), any one between the smallest and greatest is a *mean*. The most frequently used mean quantities are the *arithmetic mean*, the *geometric mean* and *harmonic mean*.

**means** The two middle terms in the proportion. In the proportion \( \frac{a}{b} = \frac{c}{d} \), for example, the means are \( b \) and \( c \).

**Mean – Value Theorem** Let a function \( f(x) \) be continuous on the interval \([a, b]\). Then there exists on \([a, b]\) a point \( c \) such that
\[
\int_a^b f(x) \, dx = f(c)(b - a).
\]

**measurement** The measurement is collection of quantitative data. A measurement is made by comparing a quantity with a standard unit. Since this comparison cannot be perfect, measurements inherently include error.

**median** Middle number of the set of data/numbers when listed in order if there is an odd number of data items, or average of two middle numbers if there is an even number of data items.

**median of a triangle** A line segment drawn from one vertex to the midpoint of the opposite side. A triangle therefore has three medians.

**mental math** When operations are performed in one’s head without other aids such as paper and pencil or calculator.

**Metamathematics** The logical analysis of mathematical reasoning.

**metric space** Metric space is a set of points such that for every pair of points there is a nonnegative real number called their distance that is symmetric and satisfies the triangle inequality.

**metric system** A system of measurement built on powers of 10.

**The Midpoint Formula** The midpoint of the line segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) has coordinates \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \). In other words, the midpoint of a line segment is equidistant from the endpoints of the line segment.

**midrange** One-half the sum of the highest and lowest scores.

**midsegment of a trapezoid** A line segment connecting the midpoints of the two nonparallel sides of a trapezoid.

**minor axis of an ellipse** The shorter axis of symmetry.
minuend The number/expression from which another number/expression (subtrahend) is being subtracted. In other words, the minuend is the number from which the subtrahend is subtracted. For example, in case $7 - 3 = 4$, the number $7$ is the minuend, $3$ is the subtrahend, and $4$ is the difference.

minute One minute, written $1'$, is $\frac{1}{60}$ of a degree.

mirror symmetry A symmetry about a plane (mirror plane) that divides the object or system into two mutual mirror images.

mixed number* A whole number and fraction written together with the understanding that they are to be added together. For example, $3 \frac{1}{2} = 3 + \frac{1}{2}$.

............

* To change a mixed number into an improper fraction, multiply the denominator by the whole number. Add the numerator. This is the new numerator. Keep the “old” denominator. For example, change $5\frac{3}{5}$ to an improper fraction. The $5$ wholes give us $5 \cdot 5 = 25$ fifths. Add these to the other $3$ fifths, or $\frac{28}{5}$.

mode The number that appears most often in a set of numbers/data. If two numbers tie for most frequent occurrence, the set/collection is said to be bimodal. Generally speaking, the set of data can be multimodal.

modeling Writing algebraic expressions or equations to represent real-life situation.

Modeling Work Problems If $a$ is the time needed for A to complete the work alone, $b$ is the time needed for B to complete the work alone, and $t$ is the time for A and B to complete the work working together, then

$$\frac{t}{a} + \frac{t}{b} = 1.$$

Example If John can paint a house in 7 hours, and Greg can paint the same house in 9 hours, how long does it take them to do it together?

Solution Let $x$ represent the number of hours for John and Greg working together. John’s working rate is $\frac{1}{7}$, and Greg’s working rate is $\frac{1}{9}$. For this problem we will make a table.

<table>
<thead>
<tr>
<th></th>
<th>Rate</th>
<th>Time</th>
<th>Part of the Job Accomplished</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>(\frac{1}{7})</td>
<td>x</td>
<td>(\frac{1}{7}x)</td>
</tr>
<tr>
<td>Greg</td>
<td>(\frac{1}{9})</td>
<td>x</td>
<td>(\frac{1}{9}x)</td>
</tr>
</tbody>
</table>

The sum of two parts of the job accomplished is 1, since one whole job is done. We must solve the following equation.
\[
\frac{1}{7}x + \frac{1}{9}x = \frac{1}{63} \quad \text{[We will multiply equation by } 63 = \text{LCD}(7,9)] \Rightarrow \quad 9x + 7x = 63 \Rightarrow \quad 16x = 63 \Rightarrow \quad x = \frac{63}{16} = \frac{3\frac{15}{16}}{\frac{16}{16}}
\]

It takes John and Greg \(3\frac{15}{16}\) hours working together to paint the house.

**Modus Ponens (“The way that affirms by affirming”)** In logic, a valid argument of the following form:

If \(p\) then \(q, p \quad \text{or} \quad p \Rightarrow q, p\)

Therefore, \(q\).

**Modus Tollens (“The way that denies by denying”)** In this law of indirect reasoning, a valid argument is of the following form:

1. If \(p\) then \(q\)
2. Not \(q\) or \(\neg q\)
3. Therefore, Not \(p\) (or \(\neg p\) or \(\sim p\)). *Note: \(p\) and \(q\) represent statements, so called premises, and \(q\) is conclusion.*

**moment** The product of the force each load exerts on a lever and its distance (or the moment arm) from the fulcrum (the 0 point). Also known as torque. Moment of the force is actually the tendency of a force to twist or rotate an object. **N. B.** The moment arm is actually the perpendicular distance from the point of rotation, to the line of action of the force.

**momentum** The product of an object’s mass and (linear) velocity. It is a vector.

**moment of inertia** A measure of the resistance of a body to angular acceleration about a given axis.

**monomial** A polynomial containing only one term.

**monotonic function** A function that is increasing or decreasing on an interval \([a, b]\) is said to be monotonic on this interval.

**monotonic sequence** A sequence \(\{a_n\}\) is monotonic when its terms are nondecreasing

\[a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots\]

or when its terms are nonincreasing

\[a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots\]

**mortgage** A debt instrument, secured by the collateral of specified real estate property, that the borrower is obliged to pay back with a predetermined set of payments*. 

………………

*To calculate the monthly mortgage payment \(M\) we will use The Monthly Mortgage Payment Formula:

\[
M = P \left[ \frac{r}{12} \left( 1 + \frac{r}{12} \right)^{12T} \right] \left( 1 + \frac{r}{12} \right)^{12T} - 1 \right]
\]

where \(P\) is the principal or the initial amount we borrowed, \(r\) is interest rate, and \(T\) is the term of mortgage in years.
motion problem A problem that deals with distance \((d)\), speed/rate \((r)\), and time \((t)\), using the formula \(d = rt\).

Example A student walks and jogs to college each day. She averages \(5 \text{ km/h}\) walking and \(9 \text{ km/h}\) jogging. The distance from home to college is 8 km, and makes the trip in 1 hr. How far does the student walk and jog? We can organize this information in a chart.

Solution We will start with distance formula and table.

\[
d = r \cdot t \quad \Rightarrow \quad t = \frac{d}{r} \quad \& \quad r = \frac{d}{t}
\]

<table>
<thead>
<tr>
<th></th>
<th>Distance</th>
<th>Speed/Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walking</td>
<td>(d)</td>
<td>5</td>
<td>Walking time  = (\frac{d}{5})</td>
</tr>
<tr>
<td>Jogging</td>
<td>(8 - d)</td>
<td>9</td>
<td>Jogging time  = (\frac{9}{8-d})</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

Walking time + Jogging time = 1, so we will have: \(\frac{d}{5} + \frac{8-d}{9} = 1 / \cdot 45\)

(We will clear fractions by multiplying by LCD \((5, 9) = 45\) \(\rightarrow \quad 9d + 5(8 - d) = 45 \rightarrow \quad 9d + 40 - 5d = 45 \rightarrow \quad 4d + 40 = 45 \rightarrow \quad 4d = 45 - 40 \quad \text{or} \quad 4d = 5\). Therefore, the walking distance is \(d = 1.25\) km, so the jogging distance is \(8 - d = 8 - 1.25 = 6.75\) km.

multiple A number obtained by multiplying a given number by any natural number. For example, 11, 22, 33, 44, 55 are the first five multiples of 11.

multiple integral The multiple integral is the special case (or type) of definite integral extended to functions of more than one variable. Multiple integration of a function in \(n\) variables: \(f(x_1, x_2, \ldots, x_n)\) over domain \(D\) is most commonly represented by nested integral signs in the reverse order of execution (the leftmost integral sign is computed last), followed by the function and integrand arguments in proper order [the integral with respect to the rightmost argument \((x_n)\) is computed last]. The domain of integration \(D\) is either represented symbolically to every argument over each integral sign or abbreviated by a variable at the rightmost integral sign:

\[
\int \cdots \int f(x_1, x_2, \ldots, x_n) \, dx_1 \cdot dx_2 \cdot \cdots \cdot dx_n
\]

D

Example 1 Evaluate the double integral of the function \(f(x, y) = 3 - x - xy^2\) over the rectangle \(D:\) \(0 \leq x \leq 1, \ 0 \leq y \leq 3\).

Solution For evaluating the double integral, we make use of the formula* for reducing a double integral to an iterated one. Namely, we then have:

\[
\int_D (3-x-xy^2) \, dx \, dy = \int_0^1 dx \int_0^3 (3-x-xy^2) \, dy.
\]

Firstly, assuming \(x\) to be constant, we compute the integral with respect to \(y\):
\[ \int_0^3 (3 - x - x y^2) \, dy = (3y - xy - \frac{xy^3}{3}) \bigg|_0^3 = 9 - 3x - 9x = 9 - 12x. \]

Then we will compute the integral of the obtained function with respect to the variable \( x \).

Therefore, \( \iint_D (3 - x - x y^2) \, dx \, dy = \int_0^1 (9 - 12x) \, dx = (9x - 6x^2) \bigg|_0^1 = 3. \)

*Formula for reducing a double integral to an iterated one*

If a function \( f(x, y) \), \( (x, y) \in D: \ a \leq x \leq b, \ c \leq y \leq d \) is continuous, then it is integrable on \( D \), and

\[ \iint_D f(x, y) \, dx \, dy = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx. \]

**Example 2** Evaluate the triple integral of the function \( f(x, y, z) = x^2 + xz + xyz \) over the parallelepiped \( D: \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1. \)

**Solution** To calculate the triple integral, we will make use of the formula for the reducing the triple integral to an iterated integral.

\[ \int \int \int_D f(x, y, z) \, dx \, dy \, dz = \int_0^1 dx \int_0^1 dy \int_0^1 (x^2 + xz + xyz) \, dz = \int_0^1 dx \int_0^1 (x^2 z + \frac{xz^2}{2} + \frac{xyz^2}{2}) \bigg|_{z=0}^{z=1} \, dy = \int_0^1 dx \int_0^1 (x^2 y + \frac{xy^2}{2} + \frac{xy^2}{4}) \, dy = \int_0^1 (x^2 + \frac{x}{2} + \frac{x}{4}) \, dx = \left( \frac{x^3}{3} + \frac{3x^2}{8} \right) \bigg|_0^1 = \frac{1}{3} + \frac{3}{8} = \frac{17}{24}. \]

**multiplicand** In multiplication, the multiplicand is the factor being multiplied.

**multiplication** The multiplication is the process of taking one number (called multiplicand) a given number of times (this is the multiplier, which tells us how many times the multiplicand is to be taken). The result is called the product. The numbers (expressions) multiplied together are called the factors of the product. In other words, multiplication is actually repeated addition.

**multiplicative identity** The number 1.

**multiplicative inverses** Other name for reciprocals (two numbers whose product is 1).

**The Multiplication Property of One** If \( a \) is a real number, then \( a \cdot 1 = 1 \cdot a = a \).

**The Multiplication Property of Zero** If \( a \) is a real number, then \( a \cdot 0 = 0 \cdot a = 0 \).

**The Multiplicity of the Zero** The number of times a zero occurs is referred to as the multiplicity of the zero. For example, 4 is a zero of multiplicity 3 of the polynomial function \( f(x) = (x - 4)^3 \), because factor \( x - 4 \) occurs 3 times.

**multiplier** In multiplication, the multiplier is the factor being multiplied by. For example, \( 3 \cdot 4 = 12 \), 3 is multiplicand, 4 is multiplier, and 12 is the product.

**mutually exclusive events** The events that cannot occur simultaneously.
**mutual funds** A mutual fund is an investment tool that enables investors to indirectly own a wide variety of stocks, bonds, or other investments.

**natural/counting numbers** The set of numbers \( \{1, 2, 3, \ldots\} \).

**natural logarithm** A logarithm with base \( e \).

**necessary condition** In conditional statement “If \( A \), then \( B \)” the consequent \( B \) is necessary condition for the antecedent \( A \).

**negation** If \( p \) is a statement, then not \( p \) (or \( \sim p \)) is its negation.

**negative angle** An angle that is formed by clockwise rotation around its endpoint/vertex.

**negative number** Any number that is less than 0.

**negative exponent** For any real nonzero number \( a \) and any integer \( n \), \( a^{-n} = \frac{1}{a^n} \) or \( (\frac{a}{b})^{-n} = (\frac{b}{a})^{n} \) (negative exponent leads to the reciprocal with positive exponent).

**negative relationship/correlation** A relationship between two sets of numerical data in which one set generally increases as the other decreases.

**net pay** Take home pay found after subtracting total and personal deductions from gross pay.

**Newton’s Law of Cooling** From experimental observations it is known that the surface temperature of an object change at a rate proportional to its relative temperature. That is the difference between its own temperature and the temperature of the surrounding environment (ambient temperature). That is what is known as Newton’s Law of Cooling. Thus, if \( T(t) \) is the temperature of the object at time \( t \) and \( T_s \) is constant temperature of the surrounding environment, then we have differential equation

\[
\frac{dT}{dt} = -k(T - T_s).
\]

Actually, Newton’s law of cooling makes a statement about an instantaneous rate of change of the temperature. The solution of this first-order linear differential equation, under initial condition \( T(0) = T_0 \), is given by

\[
T(t) = T_s + (T_0 - T_s)e^{-kt}.
\]

Hence,

\[
\frac{T(t_1) - T_s}{T(t_2) - T_s} = e^{-k(t_1 - t_2)}
\]

This equation makes it possible to find \( k \) if interval of time \( t_1 - t_2 \) is known, i.e.,
\( k \left( t_1 - t_2 \right) = -\ln \left[ \frac{T(t_1) - T_s}{T(t_2) - T_s} \right]. \)

**Example (Surrounding medium of unknown temperature)**  A pan of warm water (46° C) was put in a refrigerator. Ten minutes later, the water’s temperature was 39 °C; 10 min after that, it was 33 °C. Use Newton’s law of cooling to estimate how cold the refrigerator was.

**Solution**

\[
T_s - T_s = (T_0 - T_s)e^{-10k} \Rightarrow 39 - T_s = (46 - T_s)e^{-10k} \quad \text{and} \quad (10 \text{ min after that})
\]

\[
33 - T_s = (46 - T_s)e^{-20k} \Rightarrow 33 - T_s = (46 - T_s)e^{-10k}\quad \text{and} \quad e^{-10k} = (e^{-10k})^2 \Rightarrow
\]

\[
\frac{33 - T_s}{46 - T_s} = \frac{(39 - T_s)^2}{46 - T_s} = (33 - T_s)(46 - T_s) = (39 - T_s)^2 \Rightarrow 1518 - 79T_s + T_s^2 = 1521 - 78T_s + T_s^2 \Rightarrow
\]

\[
-T_s = 3 \quad \Rightarrow \quad T_s = -3^{\circ}C.
\]

**Newton – Leibnitz Formula**  If a function \( f(x) \) is continuous on the interval \([a, b]\), and the function \( F(x) \) is an antiderivative of \( f(x) \) on \([a, b]\), then the following formula is valid:

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

This is the Newton – Leibniz formula. This is also called the Fundamental Theorem of Calculus.

Gottfried Wilhelm Leibnitz (1646-1716) Isaac Newton (1642-1727)

**Newton’s Law of Universal Gravitation**  This law states that any two bodies in the universe attract each other with a force \( F \) that is directly proportional to the product of their masses \( m_1 \) and \( m_2 \) and inversely proportional to the square of the distance \( r \) between them.

\[
F = G \frac{m_1 m_2}{r^2},
\]

where \( G \) is the universal gravitational constant as an key quantity in this law. The value of \( G \) is approximately equal to \( 6.674 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \).

**Newton’s Method**  A technique/method for approximating a real zero \( r \) of a differentiable function \( y = f(x) \) such that \( f(r) = 0 \). If \( x_n \) is an approximation to \( r \), then the next approximation is given by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

provided \( f'(x_n) \neq 0 \).

With this special application of derivatives we can find how to approximate solutions to an equation.

**Example**  Estimate \( \pi \) to as many decimal places as your calculator will display by using Newton’s method to solve the equation \( \tan x = 0 \) with \( x_0 = 3 \).

**Solution**  \( f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow x_{n+1} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)} \); \( x_0 = 3 \Rightarrow x_1 = 3.13971 \Rightarrow x_2 = 3.14159 \) and we approximate \( \pi \) to be 3.14159.
nonhomogeneous differential equation A second-order nonhomogeneous differential equation is an
differential equation of the form
\[ y'' + p(t) y' + q(t) y = g(t), \]
where \( p(t), q(t), \) and \( g(t) \) are given continuous functions of the time \( t \) on the open interval \( I \).
The equation \( y'' + p(t) y' + q(t) y = 0 \) is the corresponding second-order homogeneous differential equation.

nonreal complex number A complex number \( a + bi \) with \( b \neq 0 \) is a nonreal complex number. For example, \( 3 + 4i \) is an nonreal complex number.

nonstrict inequality An inequality in which the symbol of inequality is either \( \leq \) or \( \geq \).

normal distribution A normal distribution is an special arrangement of a data set in which the mean, median, and mode all have the same value and all occur at the center of the distribution.

normal plane See Binormal vector

nth root A number \( b \) is called the \( n \)th root of \( a \) if \( b^n = a \).

number* A number is a mathematical object (or abstract entity) used to count and measure. In the third century B.C., Euclid defined number as a “collection made up of units”.

*1. “Number rules the universe.” 2. “Numbers have a way of taking a man by the hand and leading him down the path of reason.” (Pythagoras) 3. “Numbers are the Universal language offered by the deity to humans as confirmation of the truth”. (St. Augustine of Hippo (354 – 430))

4. “If you only knew the magnificence of 3, 6 and 9, then you would have a key to the universe.” (Nikola Tesla) 5. The number is the symbol of casual necessity.” [Oswald Spengler, German philosopher of history and mathematics, (1880 - 1936)]

number line A horizontal straight line whose points correspond with the set of real numbers in increasing order from the left to the right. In other words, every point of the number line depicts some real number or each point is the graph of a number.

number series Suppose we are given a sequence of numbers \( \{a_n\}, \ n \in \mathbb{N} \). The expression
\[ a_1 + a_2 + a_3 + \ldots + a_n + \ldots \quad \text{or} \quad \sum_{n=1}^{\infty} a_n \quad (1) \]
is called a number series, and the numbers \( a_1, a_2, \ldots, a_n, \ldots \) are the terms of the series (1).
The sums \( S_1 = a_1, \ S_2 = a_1 + a_2, \ldots, S_n = a_1 + a_2 + \ldots + a_n, \ldots \) are called partial sums of series (1).
The series \( \sum_{n=k+1}^{\infty} a_n = a_{k+1} + a_{k+2} + \ldots \) is the \( k \)th remainder of series (1).
Number of Zeros Theorem A function $f(x)$ defined by a polynomial of degree $n$ has at most $n$ distinct zeros.

numeral A conventional symbol (letter, word) that represents a number. In other words, it is a written sign depicting a number.

numerator The number above the fraction bar in a fraction, representing the number of parts of the whole (or group) that are being considered. In other words, the number indicating how many parts are taken is the numerator of the fraction.

numerical data Data consisting of numbers representing counts or measurements.

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O

objective function/quantity In linear programming, the function/quantity or expression to be maximized or minimized.

oblique asymptote A straight line $y = kx + b$ is called an oblique asymptote to the graph of a function $f(x)$ as $x \to \pm \infty$ if
\[ \lim_{x \to \pm \infty} [f(x) - (kx + b)] = 0. \]
To find the equation of the oblique asymptote we will use limits:
\[ k = \lim_{x \to \pm \infty} \frac{f(x)}{x}; \quad b = \lim_{x \to \pm \infty} [f(x) - kx]. \]
In other words, the oblique asymptote is nonvertical, nonhorizontal line that a graph approaches as $|x|$ increases without bound.
N. B. In particular case $k = 0$ the asymptote is called horizontal.

oblique triangle A triangle that is not a right triangle.

obtuse angle An angle measuring more than $90^\circ$ but less than $180^\circ$.

obtuse triangle A triangle with one obtuse angle.

octagon A polygon with eight sides. A regular octagon is a octagon with equal sides and angles. Area of a regular octagon is given by formula $A = 2a^2(1 + \sqrt{2})$, where $a$ is the length of the given side.

octant Any of the eight parts into which a coordinate space is divided by three coordinate planes (x-y, x-z and y-z plane).
**odd function** The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$, $\forall x \in D$, where $D$ is domain of $f(x)$. In addition, the graph of odd function is symmetric with respect to the origin.

**odd number** Any integer that cannot be divided evenly by 2.

**odds** A ratio of the number of ways an event can occur to the number of ways the event cannot occur. In other words, odds are the ratio of the number of possible successes to the number of possible failures.

**Odds Against an Event** The following may be used to determine the odds against an event.

\[
\text{Odds against an event} = \frac{P(\text{event fails to occur})}{P(\text{event occurs})} = \frac{P(\text{failure})}{P(\text{success})}
\]

**Odds in Favor of an Event** The following formula may be used to determine the odds in favor of an event.

\[
\text{Odds in favor of event} = \frac{P(\text{event occurs})}{P(\text{event fails to occur})} = \frac{P(\text{success})}{P(\text{failure})}
\]

**one-to-one correspondence or bijection** A function $y = f(x)$ is one-to-one correspondence or bijection if, for different $x$-values we have different $y$-values, and, conversely, i.e. $a \neq b$ implies $f(a) \neq f(b)$, and $f(a) \neq f(b)$ implies $a \neq b$. In other words, The sets $A$ (domain) and $B$ (range) are said to be put into one-to-one correspondence if to each member of the set $A$ there corresponds the only one member of the set $B$, and, conversely, to each member of the set $B$ there corresponds the only one member of the set $A$. Shortly speaking, If a function is defined so that each range element is used only once, then this function is called a one-to-one correspondence or bijection (or bijective function). The sets $A$ and $B$ that can be put into one-to-one correspondence are called equivalent; we then write $A \sim B$.

**one-to-one function** A function $y = f(x)$ is one-to-one on a domain $D$ if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in $D$.

**open interval** The set of all real numbers $x$ such that $a < x < b$ is called an open interval with $a$ and $b$ as endpoints and is denoted by $(a, b)$.

**open statement** A sentence that is neither true nor false. For example, $7x = 28$.

**opposites** Two numbers are opposites or additive inverses if their sum is 0. For example, the integers 3 and $-3$ are opposites because $3 + (-3) = 0$. In other words, opposites are the same distance from 0 on the number line but on different (opposite) sides of 0.

**opposite of a vector** The opposite of a vector $\mathbf{v}$ is a vector $-\mathbf{v}$ that has the same magnitude as $\mathbf{v}$ but opposite direction.

**optimization** Finding the greatest or least numerical value of a function as it complies with various conditions or, shortly, finding the minimum or maximum value of a quantity.
**orbital period** The time $T$ it takes a planet to go around its sun once is the planet’s *orbital period*.

**order of a differential equation** The order of a differential equation is the order of the highest derivative (or differential) of the unknown function that appears in the equation. For example, $y''' + 5e^x y'' + 2yy' = 0$ is a third order differential equation. The most general form of an $n$th order differential equation with independent variable $x$ and unknown function $y = f(x)$ is $F(x, f, f', f'', \ldots , f^{(n)}) = 0$

**order of operations (Please Excuse My Dear Aunt Sally):**

1. Do all calculations within grouping symbols (Parentheses ( ), followed by brackets [ ], and then braces { }) before operations outside.

2. Evaluate all Exponential expressions.

3. Do all Multiplications and Divisions in order from left to right.

4. Do all Additions and Subtractions in order from left to right.

**ordered pair** A pair of numbers of the form $(x, y)$ for which the order in which the numbers are listed is important.

**ordinal number** A whole number that shows the order of a unit/object in a given ordered sequence/series as first, second, third, etc.

**ordinary annuity** See annuity.

**ordinate** The second coordinate in an ordered pair/triple of numbers.

**origin** The intersection of coordinate axes. In other words, the origin is the point where the number lines cross, and this special point is the zero point on both number lines/coordinate axes.

**orthocenter of a triangle** The orthocenter of a triangle is the point where its three altitudes intersect.

**orthogonal** Intersecting or lying at right angles, or, briefly, pertaining to right angles.

**orthogonal trajectory** An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family orthogonally.

**orthogonal vectors** Orthogonal vectors are two vectors that are perpendicular, meaning that the angle between the two vectors is $90^\circ$.

**osculating plane** See Binormal vector.
**outcome** A particular result of an experiment.

**outlier** A number in a set of data that is much larger or smaller than most of the other number in the set.

**output** An element of the range of a function (an y-value).

**palindrome** A palindrome is a word, phrase, number, or other sequence of symbols or elements, whose meaning be interpreted the same way in either forward or reverse direction (Wikipedia). For example, 121 is an *palindromic number or numeral palindrome*.

**palindrome prime** A prime number that remains a prime number when its digits are reversed. For example, 13 is an palindrome prime, because 31 is also prime.

**parabola** (See ‘conic sections’) The set of all points in a plane equidistant from a fixed coplanar point (called the focus) and a fixed coplanar line (called the directrix).

**paradox** A statement that appears to be true and false at the same time.

**parallel lines** The lines in the same plane that never intersects. Two or more lines are parallel if and only if they have the same slope.

**parallelepiped** A parallelepiped is a prism whose bases are parallelograms.

**parallelogram** A quadrilateral with two pairs of parallel sides.

**parallelogram rule/method** This rule/method is a way to find the sum of two vectors. If the vectors are placed so that their initial points coincide and a parallelogram is completed that has these two vectors as two of its sides, then the diagonal vector of the parallelogram that has the same initial point as these two vectors is their sum.

**parameter** An arbitrary constant whose value characterizes a member of a system. For example, the focal parameter of the parabola (the distance between the focus and directrix of parabola). Parameter in statistics is any measured characteristic (for example, mean) of a population.

**parametric curve** If coordinates $x$ and $y$ of the points of the given curve are given as functions $x = f(t)$, $y = g(t)$ over an interval $l$ of $t$ – values, then the set of points $(x, y) = [f(t), g(t)]$ defined by these equations is a parametric curve. The given equations are parametric equations for the curve. For example, the parametric equations
\[ x = t^2, \quad y = t + 1, \quad t \in (-\infty, \infty) \]
represents horizontal parabola with Cartesian equation \[ x = (y - 1)^2. \]

**parametric equations** A parametric equations of a curve is an representation of this curve through equations expressing the coordinates of the points \((x, y)\) of the curve as functions of a third variable called parameter.

**parsec** The parsec (symbol \(pc\)) is a unit of length/distance used in astronomy. It is about 3.26 light-years, or \(1 \text{ pc} \approx 3.08567758 \cdot 10^{16} \text{ meters}\).

**partial derivatives** The derivative of the function \(f(x, y)\) with respect to the variable \(x\) at the point \((x_0, y_0)\) is called the partial derivative of the function \(f(x, y)\) with respect to \(x\) at the point \((x_0, y_0)\) and is denoted by \(f_x(x, y)\) or \(\frac{\partial f}{\partial x}\). In other words, the partial derivative of \(f\) with respect to \(x\) is the derivative of \(f\) obtained by treating \(x\) as a variable and \(y\) as a constant. In the same manner we can define the partial derivative \(f_y(x, y)\) of the function \(f(x, y)\) with respect to \(y\) (\(y\) is a variable and \(x\) is a constant) at the point \((x_0, y_0)\). Hence it follows that the partial derivatives are found by the ordinal differentiation rules for functions of one independent variable. Similar definitions could be given for the functions of more than two independent variables (arguments).

**Example** Find the \(f_x(x, y)\) and \(f_y(x, y)\) for the following function \(f(x, y) = x^2 - 3xy - 4y^2 - x + 2y + 1\).

**Solution** To find \(f_x(x, y)\) we will treat \(y\) as a constant and \(x\) as a variable.
Thus,
\[ f_x(x, y) = (x^2)' - (3xy)' - (4y^2)' - (x)' + (2y)' + (1)' = 2x - 3y - 1. \]

N. B. Derivative of a constant is 0.

Now, to find \(f_y(x, y)\), we will treat \(y\) as variable and \(x\) as a constant.
Thus,
\[ f_y(x, y) = -3x - 8y + 2. \]

**partial differential equation** A partial differential equation is an type of differential equation involving unknown function(s) of several variables and their partial derivatives.

**partial fraction decomposition** When one (algebraic) fraction (rational expression) is expressed as the sum of two or more simpler fractions, so called partial fractions, the sum is called the partial fraction decomposition.

**Example** Find the partial fraction decomposition of \(\frac{3x-1}{x^2 + x}\).

**Solution** \(\frac{3x-1}{x^2 + x} = (\text{or after factorization of the denominator}) = \frac{3x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}/x(x+1)\) ... We will multiply both sides by \(x(x + 1)\) to get
\[ 3x - 1 = A(x + 1) + Bx. \]
(1)
To find value of \(A\) we will substitute 0 for \(x\): \(3(0) - 1 = A(0 + 1) + B(0) \rightarrow 0 - 1 = A + 0 \rightarrow A = -1\)
Replace \(A\) with \(-1\) in equation (1) and substitute \(-1\) for \(x\) to get the following
\[ 3(-1) - 1 = -1(-1 + 1) + B(-1) \rightarrow -3 - 1 = -1(0) - B \rightarrow -4 = -B \rightarrow B = 4 \]
Thus, we will have: \( \frac{3x-1}{x^2+x} = \frac{-1}{x} + \frac{4}{x+1} \).

**N. B.** If the given rational expression is an improper fraction we will divide numerator by denominator, and after that we will find the partial fraction decomposition of the remainder, which is a proper fraction.

**Partial solution of the differential equation** Every solution of a differential equation obtained from the general solution of a concrete value of the arbitrary constant of integration \( C \) is called a *partial solution*.

**Pascal’s triangle** A triangular array (See *Binomial theorem*) of the binomial coefficients that occurs in expanding binomial: \((a + b)^n = a^n + na^{n-1}b + \ldots + nab^{n-1} + b^n\), named after Blaise Pascal (1623 – 1662), famous French mathematician and philosopher, although it had been discovered and described centuries earlier by famous Persian mathematician, astronomer, and poet Omar Khayyam (1048 – 1131), and Chinese mathematicians Jia Xian (c. 1050) and Yang Hui (1238 – 1298).

```plaintext
(n = 0) 1
(n = 1) 1 1
(n = 2) 1 2 1
(n = 3) 1 3 3 1
(n = 4) 1 4 6 4 1
```

**Partial sum(s) of series** See *number series*.

**Pentagon** The polygon with five sides.

**Percentage** A fraction or ratio with 100 as its denominator. The symbol for percentage, %, is a distortion of the notation \( c_{to} \) which is a contraction of the word “cento” (one hundredth).

**Percent* Another way of expressing hundredths, or a number divided by 100, usually denoted by the symbol %. In other words, a percent is a ratio with denominator 100. For example, 3% means \( \frac{3}{100} \) or 0.03.

*The expression “per cent” (from the Latin “per centum”, “per one hundred”) means a hundredth part or \( \frac{1}{100} \).
Therefore, percent is a special ratio that compares a number to a 100 using the symbol %. In other words, percent means part per one hundred.

SOLVING PERCENT PROBLEMS USING PERCENT EQUATION OR PERCENT PROPORTION

\[ \text{Amount} = \text{Percent number} \times \text{Base} \quad \text{or} \quad a = pb \quad \text{or} \quad \frac{a}{b} = \frac{p}{100} \]

In some textbooks, the percent proportion/equation is written using the words part and whole:

\[ \frac{\text{percent}}{100} = \frac{\text{part}}{\text{whole}} \quad \text{or} \quad \text{percent} \cdot \text{whole} = \text{part} \]

1. What is 20% of 40?
   
   We have to find here just amount. \[ 15 = p\% \cdot 50 \quad \text{or} \quad 15 = \frac{p}{100} \cdot 50, \] after canceling out:
   
   \[ a = 20\% \text{ of } 40 \quad \text{or} \quad a = 0.20 \cdot 40 = 8 \]

2. What percent of 50 is 15?
   
   \[ 15 = \frac{p}{2}, \] after multiplying by 2 both sides we will have: \[ 30\% = p \]

3. Five percent of what is 30?
   
   In this case, we have to find base (whole amount). Therefore, \[ 30 = \frac{5}{100} \cdot b \] \[ (5\% = 0.05 = \frac{5}{100} = \frac{1}{20}) \rightarrow 30 = \frac{1}{20} \cdot b \] or \[ 30 = \frac{b}{20}, \] and after multiplying both sides by 20 we will have \[ 600 = b. \]

HOW TO CHANGE A PERCENT TO A DECIMAL NUMBER?
   
a) Divide the number of percent by hundred; b) Remove the percent sign %.

   **Example** Change 20% to a decimal number.
   
   **Solution** \[ 20\% = 20 \div 100 = 0.2 \quad (\% \text{ means } \frac{1}{100}) \]

HOW TO CHANGE A FRACTION TO A PERCENT?

   a) Divide the numerator by the denominator;
   b) Multiply the quotient by 100 (which has effect of moving the decimal point two places to the right);
   c) Add percent sign %.

   **Example** Change \( \frac{5}{8} \) to a percent.
   
   **Solution** \[ \frac{5}{8} = 0.625 = 62.5\% \]

HOW TO CHANGE A DECIMAL NUMBER TO A PERCENT?

   a) Multiply decimal by 100 and add percent sign %.

   **Example** Change 0.7324 to a percent.
   
   **Solution** \[ 0.7324 = 0.7324 \times 100 = 73.24\% \]

percentiles (or centiles) The numbers that divide a set of data into 100 equal parts. Actually, the 99 percentiles divided the ranked data into 100 groups with 1% of the scores in each group. Percentile (or centile) is the value of a variable below which a certain percent of observation falls. In other words, percentiles will tell us what percent of a set/group of numbers is less than or equal to a given number. The 25th percentile is also known as the first quartile \( (Q_1) \), the 50th percentile as the median or second
quartile \((Q_2)\), and 75\textsuperscript{th} percentile as the third quartile \((Q_3)\).

**Example** One College Algebra class had the following raw scores on the test: 121, 84, 111, 83, 85, 120, 162, 150, 114, 180, 155, 96, 124, 135, 172, 156, 153, 89, 91, 158, 165, 168, 119, 123, 148. John’s score was 153. What was his percentile ranking?

**Solution** To find John’s percentile ranking, follow these steps:
1) Arrange the data in order from least to greatest: 83, 84, 85, 89, 91, 96, 111, 114, 119, 120, 121, 123, 124, 135, 148, 150, 153, 155, 156, 158, 162, 165, 168, 172, 180.
2) Find the total number of scores: \(n = 25\).
3) Find how many scores are less than or equal to John’s score. We can see that 16 of the 25 scores are less than or equal to 153.
4) Find out what percent 16 is of 25?
\[
\frac{16}{25} = 0.64 = 64\%
\]
Therefore, 64\% of the scores are less than or equal to John’s score. So John’s score is in the 64\textsuperscript{th} percentile.

**perfect cube** When a whole number or fraction is multiplied three times by itself (cubed) the number obtained is called a perfect cube. For example, 27 is a perfect cube because \(3^3 = 3 \cdot 3 \cdot 3 = 27\).

**perfect number** A positive integer (natural number) that is the sum of its proper positive divisors excluding the number itself. The first four perfect numbers are \(6, 28, 496\), and \(8,128\). It is not known if any odd perfect number exists.

**perfect square** When a whole number or fraction is multiplied by itself (squared) the number obtained is called a perfect square.

**perfect square trinomial** A trinomial that factors into square of a binomial. For example, \(a^2 + 2ab + b^2 = (a + b)^2\) is an perfect square trinomial.

**perfect triangle** A triangle whose sides have whole number lengths and whose area is numerically equal to its perimeter. For example, the triangle with sides of length 9, 10, and 17 is perfect, because its area and perimeter are both 36.

**perimeter (P)** The distance around a closed plane figure. In other words, this is the sum of the sides of the figure.

**period of the function** The period or cycle of cyclic/periodic function measures how long it takes to complete one cycle of the function.

**Periodicity of Functions** A function \(y = f(x), x \in A\), is called periodic if there exists a number \(P \neq 0\) such that for any \(x\) from the set \(A\) (domain) and every integer \(n\) the equality \(f(x) = f(x + nP)\) is satisfied. In this case, the number \(P\) is called the period of the given function \(f\).

**periods** Groups of three digits, separated by commas.
**permutation** Any ordered arrangement of a given set of objects. The number of permutations of \( n \) distinct items is given by formula \( P(n) = n! \), where \( n! \) (n factorial) is product of the first \( n \) natural numbers. The number of permutations possible when \( r \) objects are selected from \( n \) objects is given by the formula

\[
\text{nPr} \ [\text{or } P(n,r)] = \frac{n!}{(n-r)!}
\]

**Example** In how many ways can 8 of 12 monkeys be arranged in a row for a genetics experiment?

**Solution** Since the order here does not matter, we have here the permutation of 12 elements taken 8 at a time. Therefore,

\[
P(12,8) = \frac{12!}{(12-8)!} = \frac{12!}{4!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 19,958,400.
\]

There are just 19,958,400 different ways to arrange 8 of 12 monkeys.

**Permutations of Duplicate Objects** The number of **distinct permutations** of \( n \) objects where \( n_1 \) of the objects are identical, \( n_2 \) of the objects are identical, \ldots, \( n_r \) of the objects are identical is given by the formula

\[
\text{\(n!\) \(n_1!n_2!\ldots n_r!\).}
\]

**Example** Find the number of distinguishable permutations of the letters in word *infinity*.

This word contains 3 i’s, 2 n’s, 1 f, 1 t, and 1 y.

**Solution** To use the formula, let \( n = 8 \), \( r = 5 \) (five different letters), \( n_1 = 3 \), \( n_2 = 2 \), \( n_3 = 1 \), \( n_4 = 1 \), and \( n_5 = 1 \).

\[
\frac{n!}{n_1!n_2!n_3!n_4!n_5!} = \frac{8!}{3!2!1!1!1!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!2!1!1!1!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2 = 3360
\]

There are 3360 distinguishable permutations of the letters in the given word.

**perpendicular lines** The straight lines that form a right angle. Two lines are perpendicular if and only if their slopes are opposite reciprocals, or, in other words, if the product of their slopes is \(-1\). For example, \( \frac{2}{5} \) and \( -\frac{5}{2} \).

**phase shift** For periodic function, a horizontal translation is a phase shift.

**pi (\( \pi \)**) An transcendental/non-algebraic number. It is the ratio of the circumference of a circle to its diameter, i.e. \( \pi = \frac{C}{2r} \approx 3.14159265358979 \ldots \) It is also sometimes called Archimedes' constant or Ludolph's constant. The symbol \( \pi \) was first used by Welsh mathematician **William Jones** in 1706, and subsequently adopted by **Euler**. In “Measurement of a circle” **Archimedes** (ca. 225 BC) obtained the first rigorous approximation*: \( 3 + \frac{10}{71} < \pi < 3 + \frac{1}{7} \). Famous Chinese mathematician, astronomer and mechanic **Zu Chongzhi** (429 – 500) took credit for three approximations of \( \pi \): \( \frac{22}{7} \), \( \frac{355}{113} \), and the interval \( 3.1415926 < \pi < 3.1415927 \). The third result he found after calculating more than 1000 times and it remained the best in the world until improved by the Arab mathematician **al-Kashi** (1400). Zu Chongzhi also yields methods** of cubic equations and he worked on deducing the formula for the volume of a sphere.

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*The Bible contains two references (I Kings 7: 23 and Chronicles 4: 2) which give a value of 3 for \( \pi \) (See: [http://www.mathworld.wolfram.com/Pi.html](http://www.mathworld.wolfram.com/Pi.html)).

**Leibniz formula for \( \pi \)** The Leibniz formula for \( \pi \) named after Wilhelm Gottfried von Leibniz states that

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \text{(using summation notation)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.
\]

**photon** A photon is an elementary particle the quantum of light and all other forms of electromagnetic radiation.

**piecewise-defined function** A function that is collection of two or more various functions over different parts of its domain.

**place value** A position of a digit in a number or worth of a digit based on its position in a numeral.

**place-value system** The position or placement of the digit in a number tells the value of the digit. For this reason our number system is called a place-value system.

**Plank-Einstein equation** Plank-Einstein equation describes the relationship between energy \( E \) and frequency \( \nu \) of an photon:

\[
E = h \nu, \quad \text{where } h \text{ is Planck's constant, } h = 6.62606957 \times 10^{-34} \frac{m^2kg}{s}.
\]

**Platonic/Regular Solids** A Platonic solid is polyhedron whose faces are all regular polygons of the same size and shape. There are five Platonic solids: Tetrahedron (4 faces, 4 vertices, 6 edges), Cube (6 faces, 8 vertices, 12 edges), Octahedron (8 faces, 6 vertices, 12 edges), Dodecahedron (12 faces, 20 vertices, 30 edges), Icosahedron (20 faces, 12 vertices, 30 edges). Platonic solids, also called regular polyhedra, are named after the ancient Greek philosopher Plato.

**point** An undefined term in geometry, like other basic terms (line segment, straight line, plane,...).

Euclid* left us his definition (ca 300 BC) of the point: “A point is that which has no parts.” For a possible
discoverer and great practitioner of the axiomatic method, he is strangely oblivious about one-step logical implication of his definition of the point. What is “that” in his definition of the point he did not say.

*Euclid: “The laws of nature are but the mathematical thoughts of God.”

**Point-slope form of the equation of the straight line** An linear equation, described by $y - y_1 = m(x - x_1)$, where $m$ is the slope and $(x_1, y_1)$ is a given point that lies on the graph of the equation/straight line.

**Polar coordinate system** A coordinate system based on the origin (the pole), polar axis (positive x-axis), and directed angle $\theta$. The position of an arbitrary point $P$ in the plane will be defined if a pair of numbers $r$ and $\theta$ corresponding to this point are given, where $r$ is the directed distance from the origin $O$ to the point $P$, and $\theta$ the directed angle from positive x-axis (the polar axis) to ray $OP$. The number $r$ and $\theta$ are called the polar coordinates of the point $P$. If the point $P$ is given by the polar coordinates $r$ and $\theta$, then we write $P(r, \theta)$.

**Polar equation** An equation that uses polar coordinates. The variables (unknowns) are $r$ and $\varphi$.

**Pole** The pole is single fixed point in the polar coordinate system that is the endpoint of the polar axis. The pole is usually placed at the origin of a rectangular coordinate system.

**Polygon** A closed plane figure determined by three or more line segments as sides.

**Polyhedron** *(plural, polyhedra)* A 3-dimensional figure/solid bounded by plane surfaces called faces/polygons. The faces intersect in line segments called edges. The point at which two or more edges intersect I called a vertex.

*Euler’s Polyhedron Formula*

$\text{Number of vertices} - \text{number of edges} + \text{number of faces} = 2$

**Population** A group of people (or animals or objects or events) that fit a particular description. Generally, this is the complete and entire collection of elements to be studied.

**Population paradox** This paradox may occur when the population of one or more groups/states changes. Generally, the population paradox occurs when group A loses items to group B even though group A’s population grew at a faster rate than group B’s.

**Position vector** A vector with its initial point at the origin in a Cartesian (rectangular) coordinate system.

**Positive angle** A positive angle is an angle that is formed by counterclock-wise rotation around its endpoint.
**positive number** Any number that is greater than zero.

**positive relationship/correlation** For a set of data in two variables, the variables have a positive relationship if one increases as the other increases.

**postulate** A statement that is accepted as true without proof.

**power** The power $a^n$ of the number $a$ $(a \in R, n \in \mathbb{N})$ is the product of $n$ factors each of which is equal to $a$. In the other words, that is the result of repeated multiplication.

For example, in the expression $3^2 = 9$, 9 is the second power (square) of 3, $2$ is the exponent of the power and 3 is the base.

**power function** The function $y = a^x$ $(a, n$ are constants) is called a power function.

**Power Rules:**
1. $a^n \cdot a^m = a^{m+n}$ (The Product Rule);
2. $a^m \div a^n = a^{m-n}$ (The Quotient Rule);
3. $a^{-n} = \frac{1}{a^n}$ \quad (Negative exponents);
4. $a^0 = 1, a \neq 0$ \quad (0 as an exponent);
5. $(a^m)^n = a^{mn}$ (Power Rule);
6. $(ab)^n = a^n b^n$ (Raising a product to a power);
7. $(\frac{a}{b})^n = a^n \div b^n$ (Raising a quotient to a power);
8. $(x^n \cdot y^m)^r = x^{nr} y^{mr}$.

**power series** A series of the form
$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n + \ldots,$$
where $x$ is a variable and $\{a_n\}$ is a sequence of coefficients, and $x_0$ is a constant, is called a power series about $x_0$ (or centered at $x_0$). **Power series about $x = 0$** is a series of the form
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

**preimage** A shape that undergoes a motion or transformation.

**premise** A statement which is given or accepted as being true and that is used as the basis of an argument.

**present value** The present value $P$ is the certain amount of money we must deposit in an account today to have a certain amount of money $A$ in the future. Following is a formula for determining the present value.

$$P = \frac{A}{(1 + \frac{r}{n})^{nt}},$$

where $P$ is the present value, or principal to invest now, $A$ is the amount to be accumulated in the account, $r$ is annual interest rate as decimal number, $n$ is the number of compounding periods per year, and $t$ is the time in years.

**Example** Use the present value formula to determine the amount to be invested now. The desired accumulated amount is $100,000 after 4 years invested in an account with $4\%$ interest compounded quarterly.

**Solution** To answer this question, we will use the present value formula with $A = 100,000$, $r = 0.04$, $n = 4$, $t = 4$. We have: $P = \frac{100,000}{(1 + \frac{0.04}{4})^{4 \times 4}}$.
n = 4, and t = 4.

\[
P = \frac{100,000}{(1 + 0.04)^4} = \frac{100,000}{(1 + 0.01)^{16}} = \frac{100,000}{1.01^{16}} = $85,282.13
\]

We need to invest approximately $85,282.13 now to have $100,000 in 4 years.

**present value of money flow** If \( f(x) \) is the rate of continuous money flow at an interest rate \( r \), compounded continuously, for \( t \) years, then the present value is

\[
P = \int_0^t f(x) e^{-rx} \, dx.
\]

**prime number** A whole number greater than 1 that can be divided evenly only by itself and 1. The number 2 is the only even prime number. Note that the number 1 is not considered a prime number. There are 783 prime numbers/primes in the first 6,000 natural numbers.

*Historical vignettes of mathematics make mathematics interesting. When presenting prime numbers we have to remark French Minimite monk and mathematician **Marine Mersenne** (1528-1648) and his well known formula. He claimed that numbers of the form \( 2^n - 1 \) are prime if \( n \) is prime. One of the largest known prime today is 44th Mersenne prime. The discovery was made at September, 2006 by Drs. Steven Boone and Curtis Cooper, professors at Central Missouri State University. This number in exponential form is \( 2^{32,582,657} - 1 \). The new Mersenne prime has 9,808,358 digits. Their team found also 43th Mersenne prime (February, 2005). Two years later (August 23, 2008) **Edson Smith** found 46th Mersenne prime, 12,978,189 digits long. This number in exponential form is \( 2^{43,112,609} - 1 \). To help imagine the size of the 46th known Mersenne prime, it would require 3,461 pages to display the number in base 10 with 75 digits per line and 50 lines per page. It is very interesting to know that current record holder was followed 14 days later (9/6/2008) by a smaller Mersenne, discovered by **Hans-Michael Elvenich**. 45th Mersenne prime is 11,185,272 digits long and has form \( 2^{37,156,667} - 1 \). **Curtis Cooper** on January 7, 2016 discovered the largest known prime number, \( 2^{74,207,281} - 1 \). It has 22,338,618 digits.

**Leonhard Euler** (1707-1783), one of the most prolific mathematicians who ever lived, left us wonderful observation about prime numbers: *“Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.”* By the way, there are 783 prime numbers in the first 6000 natural numbers.

**1.** "I’ll tell you once, and I’ll tell you again. There’s always a prime between \( n \) and \( 2n \)."

**2.** “God may not play dice with the universe, but something is going on with the prime numbers.”

**Paul Erdos** (1913 – 1996), Hungarian mathematician. He was one of the most prolific mathematicians of the 20th century. He pursued and proposed problems in discrete mathematics, graph theory, number theory, set theory, and probability theory. His death came only hours after he solved a geometry problem in a conference in Warsaw. Erdos was well known for his social practices of mathematics.
Namely, he engaged more than 500 collaborations, and published around 1,500 mathematical papers during his life time (Wikipedia).

**prime factorization** A composite number that is expressed as the product of its prime factors (prime numbers). For example, the prime factorization of 18:

\[ 18 = 2 \cdot 3 \cdot 3. \]

**principal (P)** An amount of money that is invested or borrowed/loaned.

**principal square root** The nonnegative square root of number.

**principle of moments** When several parallel forces act on an object, it will be in balance if the sum of the moments is 0.

**principal of squaring** If an equation \( a = b \) is true, then the equation \( a^2 = b^2 \) is true.

**prism** A prism is a polyhedron (3-dimensional solid) with two parallel bases that are congruent polygons with corresponding sides parallel, and all other faces are parallelograms.

**probability** A number from 0 to 1 (or from 0 to 100\%) that indicates how likely an event is to occur. In other words, probability is the measure of how likely an event is. The probability of an event happening is the quotient/ratio of the total number of successful events and total number of possible events.

\[
P(event) = \frac{\text{number of successes}}{\text{total number of possibilities}}
\]

**probability distribution** A listing of all the outcomes of an experiment and the probability of each of these outcomes. The sum of the probabilities in a probability distribution must always equal 1: \( P(A) + P(\text{not } A) = 1 \). The probability of an event that cannot occur is 0. The probability of an event that must occur is 1. Every probability is a number between 0 and 1 inclusive; \( 0 \leq P(E) \leq 1 \). The sum of the probabilities of all possible outcomes of an experiment is 1.

**Probability of A and B** To determine the probability of \( A \) and \( B \), we will use the following formula.

\[
P(A \text{ and } B) = P(A) \cdot P(B)
\]

**Probability of A or B** To determine of probability of \( A \) or \( B \), we will use the following formula.

\[
P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)
\]

**Probability of an Event Happening At Least Once** The probability of an event happening at least once can be determined by the following formula.

\[
P(\text{event happening at least once}) = 1 - P(\text{event does not happen})
\]

**Problem Solving** See “Four-step process for Problem Solving”.
**producers' surplus** If $S(q)$ is a supply function with equilibrium price $p_0$ and equilibrium supply $q_0$, then

\[
\text{Producers' surplus } = \int_0^{q_0} [p_0 - S(q)] \, dq.
\]

N. B. $p_0 = S(q_0)$

**product** The result when one number is multiplied by another. The numbers being multiplied are called *factors*. The first factor is also called *multiplicand* and the second factor is also called *multiplier*.

**profit** The profit function is defined as $P(x) = R(x) - C(x)$, where $R(x)$ is the revenue function, and $C(x)$ is the cost function.

**Example** Assume that the total revenue received from the sale of $x$ items is given by $R(x) = 30\ln(2x + 1)$, while the total cost to produce $x$ items is $C(x) = \frac{x^2}{2}$.

**Solution** Find the number of items that should be manufactured so that profit, $P(x) = R(x) - C(x)$, is a maximum. The profit function will be a maximum when the derivative of the profit function is equal to 0.

\[
P(x) = R(x) - C(x) = 30\ln(2x + 1) - \frac{x^2}{2};
\]

\[
P'(x) = \frac{30}{2x+1} \cdot 2 - \frac{1}{2}, \quad \text{N. B.} \quad (\ln u)' = \frac{1}{u} \cdot u_x'
\]

\[
P'(x) = \frac{60}{2x+1} - \frac{1}{2}; \quad P'(x) = 0 \rightarrow \frac{60}{2x+1} = \frac{1}{2} \rightarrow 2x + 1 = 120 \quad \text{Proportion property: If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc)
\]

So, $x = \frac{119}{2} = 59.5$.

Thus, the maximum profit occurs when $x = 59.5$ or, in a practical sense, when 59 or 60 items are manufactured. Both 59 and 60 give the same profit.

**proof** A convincing argument that uses logic to show that a statement is true.

**proper divisor** A proper divisor of a number is a positive factor of that number other than number itself.

**proper fraction** A fraction in which the numerator is smaller than the denominator.

**proportion** An equation stating that two ratios/rates/fractions are equal. For example, $\frac{a}{b} = \frac{c}{d}$, where $a$, $b$, $c$, and $d$ are non-zero real numbers, $a$ and $d$ are *extremes of the proportion*, and $b$ and $c$ are the *means of the proportion*.

**proportion property** If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

**proportional sides** The sides of two polygons are proportional if all of the ratios of the corresponding sides are equal. Generally, two mutually dependent quantities are termed *proportional* if the ratio of their values remains constant.
**proportionality**  A algebraic relation between two variables/quantities whose ratio is constant. Proportionality indicates that two variables/quantities $x$ and $y$ are related in a linear manner, i.e. $y = kx$, where $k$ is constant of proportionality.

**property of additive inverses**  For each real number $a$, there is one and only one real number $-a$ such that $a + (-a) = 0$.

**property of multiplicative inverses**  For each nonzero real number $a$ there is one and only one real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$.

**protractor** A tool used for measuring angles. It usually has $0^\circ$ through $180^\circ$ marked along a half-circle.

**purchase price** The price of an item before sales tax is added.

**pure imaginary number** A complex number $a + bi$ in which $a = 0$ and $b \neq 0$ is called pure imaginary number. For example, $4i$ is pure imaginary number.

**pyramid** A 3-dimensional figure/solid whose base is a polygon and each of whose lateral triangular faces meet at a common point called the vertex of the pyramid.

**Pythagorean theorem**  For any right triangle, the sum of the squares of the lengths of the legs, $a$ and $b$, equals the square of the length of the hypotenuse, $c$, i.e. $a^2 + b^2 = c^2$. The theorem is credited to Pythagoras of Samos, a famous Greek philosopher who lived in 6th and 5th centuries B. C (c. 570 BC – c. 495 BC). Actually this theorem was known in the Ancient East (ancient Babylonians) 20 centuries before the Christian era.

Pythagorean triple A set of three natural numbers $a$, $b$, and $c$ such that $a^2 + b^2 = c^2$.

**Q, E. D.** Quod erat demonstrandum (Lat): Which was to be proved/demonstrated. This abbreviation is used in mathematical proofs to show that what was to be proven has been proven.
quadrants  The quadrants are the four regions into which the \( x \ – \ axis \) and \( y \ – \ axis \) divide the coordinate plane.

quadrantal angle  An angle that, when placed in standard position, has its terminal side along the \( x \)-axis or \( y \)-axis. Angles with measures \( 90^\circ, 180^\circ, 270^\circ \), and so on, are quadrantal angles.

quadratic  An equation/polynomial in which the highest power of the unknown quantity/variable is a square.

quadratic/second degree equation  An equation of the form: \( ax^2 + bx + c = 0, a \neq 0 \) (standard/general or implicit form).

quadratic formula  The quadratic formula \( x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) is a general formula that can be used to solve any quadratic equation (second-degree equation) in general/implicit form \( ax^2 + bx + c = 0 \).

This formula indicates that in the solution of the quadratic equation three cases are possible:

1. \( b^2 - 4ac > 0 \); two solutions (or roots or \( x \)-intercepts or zeros) are real and distinct;

2. \( b^2 - 4ac = 0 \); one real solution of multiplicity 2 (the same solution/root/zero appears two times) or so called twofold solution;

3. \( b^2 - 4ac < 0 \); both roots/solutions/zeros are imaginary. In other words, solutions are complex numbers (conjugates).

The expression \( b^2 - 4ac \) which permits us to discriminate between the three cases is termed the discriminant (\( D \)) of quadratic equation/function.

quadratic functions and their graphs  A quadratic function is a second-degree polynomial function. The general or explicit form is \( f(x) = ax^2 + bx + c \), where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \). The graph of quadratic function is a parabola (See: ‘Conic sections’). Parabolas are U-shaped and symmetric with respect to a vertical line known as parabola’s axis of symmetry. The point of intersection of the parabola with its axis is termed the vertex of the parabola. If quadratic function has an equation in so called standard (or vertex) form \( f(x) = a(x - h)^2 + k \), then the parabola has the vertex at \( V(h, k) \), and

\[
\begin{align*}
 h &= -\frac{b}{2a}, \\
 k &= c - \frac{b^2}{4a} = -\frac{D}{4a} \\
 or \\
 V &= \left[ -\frac{b}{2a}, f\left( -\frac{b}{2a}\right) \right].
\end{align*}
\]

The parabola opens upward when \( a > 0 \) and downward when \( a < 0 \).

To graph a quadratic function: 1. Determine whether the parabola opens upward or downward. 2. Find the vertex of the parabola. 3. Find the \( x \)-intercepts. 4. Find the \( y \)-intercept. 5. Identify one additional point on the graph. 6. Draw a smooth curve through the points found in Steps 2-5.

quadric surface/second-order surface/quadric  A quadric surface is the graph in space of a second degree equation in \( x, y, \) and \( z \):
\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0. \]
The basic quadric surfaces are spheres, ellipsoids, paraboloids, elliptical cones, and hyperboloids.

**quadrilateral** A polygon with four sides.

**quartic polynomial** A polynomial in one variable of degree 4.

**quartiles** Along with the median, the quartiles divide an ordered set of data into four groups of about the same size. Generally speaking, the three quartiles divide the ranked data into four groups with 25% of the scores in each group.

**quartile** The values in a collection of numbers that separate the data into four equal parts.

**quinary number (counting) system** A five-based number system.

**quincunx** An arrangement of the five objects/points with one at each corner of a rectangle or square and one at the center (or in the middle).

**quintic polynomial** A polynomial in one variable of degree 5.

**quotient** The number/expression/quantity resulting from division of one number/expression/quantity by another.

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**R**

**radian measure of angles** A radian is a unit of measure for angles. An angle with vertex at the center of a circle that intercepts an arc on the circle equal in length to a radius of the circle has a measure of 1 radian. Radian measure allows us to treat the trigonometric functions as functions with domains of real numbers, rather than angles, because in radian measure of angles, the value of the angle is measured by a pure number.

**radical/irrational equation** An equation in which a variable appears in a radicand (See: *Irrational equations*).

**radical expression** An algebraic in which a radical sign appears.

**radical function** A function that is described by a radical expression.

**radical notation for** \( a^{\frac{m}{n}} \) If \( a \) is a real number, \( m \) is an positive integer, \( n \) is a positive integer, and \( \sqrt[n]{a} \) is a real number, then

\[
\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}.
\]
radicand The number or expression written under radical. For example in radical expression $\sqrt{3a}$ the radicand is 3a.

radius The distance from the center of a circle to a point on the circle; a line segment connecting a point on the circle to the center of the circle.

random By chance, with no outcome any more likely than another.

random sample A sample in which every person, object, or event in the population has an equal chance of being selected for the sample.

random variable A variable whose value is subject to variations due to chance (i.e. randomness, in mathematical sense). As opposed to other mathematical variables, a random variable conceptually does not have a single, fixed value (even if unknown); rather it can take on a set of possible different values each with a associated probability (Wikipedia). Or shortly, a random variable is a function that assigns a real number to each outcome of an experiment.

range The difference between the greatest and least in a set of data; it indicates the total spread of the data. The range, standard deviation, and the variance are the measures of dispersion.

Range = highest value – lowest value

range of a function/target set The set of all second coordinates (y-values) of the ordered pairs in a function, or the set of all outputs. In other words, what actually comes out of a function is called the range.

rank The numerical position of an item in a sample set arranged in order. Data are ranked when they are arranged according to some criterion, such as smallest to largest or best to worst.

rare numbers The numbers, which give a perfect square on adding as well as subtracting its reverse. The first rare number is 65 (its reverse is 56), because $65 + 56 = 121 = 11^2$ and $65 - 56 = 9 = 3^2$. The set of all rare numbers is the subset of the set of all natural numbers.

rate A ratio/quotient used to compare two different kinds of measure. In other words, rate of $a$ to $b$ is quotient $\frac{a}{b}$ of two quantities $a$ and $b$ that are measured in different units. The rate is also another term for velocity or speed.

ratio The ratio of $a$ to $b$ is $\frac{a}{b}$, also written $a : b$. In other words, ratio of two numbers/quantities is the comparison these two numbers/quantities by division. The dividend $a$ is called the antecedent of the ratio, the divisor $b$, the consequent.

rational equation An equation containing one or more rational expressions/algebraic fractions.
rational exponent  An exponent in form of an rational number/fraction. The expression with rational exponent can be expressed in form of the root, i.e. \( a^{\frac{m}{n}} = \sqrt[n]{a^m} \).

rational expressions* (or algebraic fractions) Fractions whose numerators and denominators are polynomials.

* Determining When a Rational Expression Is Undefined
We know that the denominator of the rational expression (or algebraic fraction) cannot equal 0 because division by 0 is undefined.

Step 1 Set the denominator of the given rational expression equal to 0.
Step 2 Solve this equation.
Step 3 (Final step) The solution of the given equation are the values that make the rational expression undefined. The variable cannot equal these values.

Example Find any values of the variable \( x \) for which the rational expression \( \frac{x - 2}{2x - 5} \) is undefined.

Solution  Step 1 Set the denominator equal to 0: \( 2x - 5 = 0 \);
Step 2 Solve this equation.
\[ 2x = 5 \rightarrow x = \frac{5}{2} \]
Step 3 The given rational expression is undefined for \( x = \frac{5}{2} \), so \( x \neq \frac{5}{2} \).

rational function  A function of the form \( f(x) = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomial functions.

The domain of such a function is entire number line (set of all real numbers), except for the points (numbers) at which the denominator vanishes.

rationalizing the denominator  A procedure for finding an equivalent and simpler expression without a radical(s) in the denominator. For example, \( \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} \).

rational inequality  An inequality containing a rational expression(s).

rational number  A number that can be written in the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers, and \( b \neq 0 \).

The set of rational numbers is a dense set of numbers. By this we mean that between any two fractions, no matter how close, we can always squeeze another. One of the most momentous events in the history of mathematics was the discovery that the rational numbers, despite their density, leave “holes” along the number line – points that do not correspond to rational numbers.

Rational Zeros Theorem  If \( \frac{p}{q} \) is a rational number written in lowest terms, and if \( \frac{p}{q} \) is a zero of polynomial function \( f(x) \) with integer coefficients, then \( p \) is a factor of the constant term (last term) \( a_0 \) and \( q \) is factor of the leading coefficient \( a_n \), where \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \).

raw data  Unorganized data.
ray  A part of a straight line that starts at a point and extends infinitely in one direction.

real number  A number that is either rational or irrational. Set of all real numbers (\( R \)) is union of sets of the rational (\( Q \)) and irrational (\( I \)) numbers, i.e. \( R = Q \cup I \).

real number line  A horizontal line that pictures real numbers as points.

real part  In the complex number \( a + bi \), \( a \) is called the real part of the complex number.

reason*  A statement offered in explanation or justification.

Pythagoras (570 BC, Samos Island – 495 BC, Metapontum): “Reason is immortal, all else mortal.”
Gottfried Wilhelm Leibnitz (1646 – 1716): “There is nothing without reason.”
George W. Hegel (1770-1831): ”Whatever is reasonable is true, and whatever is true is reasonable.”

*1) All reasoning has a PURPOSE.
2) All reasoning is an attempt to FIGURE something out, to settle some QUESTION, Solve some PROBLEM.
3) All reasoning is based on ASSUMPTIONS.
4) All reasoning is done from some POINT OF VIEW.
5) All reasoning is based on DATA, information, AND evidence.
6) All reasoning is expressed through, and shaped by, CONCEPTS and IDEAS.
7) All reasoning contains INFERENCES or INTERPRETATIONS by which we draw CONCLUSIONS and give meaning to data.
8) All reasoning leads somewhere or has IMPLICATIONS and CONSEQUENCES.


reciprocal  A multiplicative inverse; two numbers are reciprocals if their product is 1. For example, the reciprocal of a nonzero number \( x \) is \( \frac{1}{x} \), and \( x \cdot \frac{1}{x} = 1 \).

reciprocal function  The function defined by \( f(x) = \frac{1}{x} \) is called the reciprocal function.

reciprocals  Two numbers whose product is 1. For example, \( \frac{a}{b} \) and \( \frac{b}{a} \) are reciprocals because their product is 1.

rectangle  A parallelogram with four right angles.

rectangular form (standard form) of a complex number  The rectangular form (or standard form) of a complex number is \( a + bi \), where \( a \) and \( b \) are real numbers.

rectifying plane  See Binormal vector.
recursive definition of an sequence A sequence is defined by a recursive definition if each term after the first term or first few terms is defined as an expression involving the previous term or terms.

recursive formula A formula that is used to determine the next term of a sequence using one or more of the preceding terms.

Example Find the first four terms of the sequence $a_1 = -2$, $a_n = a_{n-1} - 3$, if $n > 1$.
Solution We know $a_1 = -2$, and we will use a recursive formula $a_n = a_{n-1} - 3$.

\[
\begin{align*}
  a_1 &= -2 \\
  a_2 &= a_1 - 3 = -2 - 3 = -5 \\
  a_3 &= a_2 - 3 = -5 - 3 = -8 \\
  a_4 &= a_3 - 3 = -8 - 3 = -11
\end{align*}
\]

reducing (simplifying) a fraction Dividing the numerator and denominator by common factor (or factors)

reductio ad absurdum (Lat.) A form of argument in which an assumption is shown to lead to an absurd or impossible conclusion so that the assumption must be rejected; literally “reduction to the absurd.”

reference angle The reference angle for an non-acute angle $\theta$ is the positive acute angle $\theta'$ made by the terminal side of the given angle $\theta$ and the x-axis. N. B. The reference angle is always found with reference to the x-axis.

Example Find the reference angle for the angle $\theta = 1391^\circ$.
Solution Firstly, we will find a coterminal angle between $0^\circ$ and $360^\circ$ by dividing $1391^\circ$ by $360^\circ$.

$1391 \div 360 \approx 3.864 \Rightarrow 1391^\circ - 3 \cdot 360^\circ = 1391^\circ - 1080^\circ = 311^\circ$

The reference angle for $311^\circ$ (and thus for $1391^\circ$) is $\theta' = 360^\circ - 311^\circ = 49^\circ$.

reference arc The reference arc for a point on the unit circle is the shortest arc from the point itself to the nearest point on the x-axis.

reflection A reflection, taking reflection line (or axis of reflection) $l$ into consideration, is a rigid and isometric motion a geometric figure to a new position on other side of the reflection line.

Reflexive Property of Equality For any real number $a$, $a = a$.

region A part of a plane.

regular polygon* A polygon in which all sides are congruent and all angles are congruent.

\[ A = \frac{s^2 n}{4 \tan \left( \frac{\pi}{n} \right)} \], where $s$ is the length of the side, and $n$ is the number of sides.

regular polyhedron A polyhedron in which all faces are congruent and the same number of polygons meet at each vertex.
relation A correspondence between two sets, such that each member of the first set (domain) corresponds to at least one member of the second set (range). In other words, an relation is a set of ordered pairs.

remainder The whole number left after one number is divided into another number.

Remainder Theorem If the polynomial \( f(x) \) is divided by \( x - k \), then the remainder is equal to \( f(k) \).

Remainder Theorem The limits \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \) and \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e \) are customarily called the first and the second remarkable limit owing to their importance in many applications in area of the mathematical analysis.

repeating decimals A decimal in which one or more digits repeat. For example, \( 0.333\ldots = 0.\overline{3} \)

repeated root A root corresponding to a factor that appears two or more times in a factorization. For example, in the equation \( (x - 3)^2 = 0 \), \( x = 3 \) is a repeated root (or solution) of the given quadratic equation, and we will say also that \( 3 \) is root of multiplicity two.

resultant If \( a \) and \( b \) are vectors, the vector sum \( a + b \) is called the resultant of vectors \( a \) and \( b \).

revenue The revenue function for selling a product depends on the price per item \( p \) and the number of items/units sold \( q \), and this function is given by \( R(q) = pq \).

rhombus A parallelogram with all sides congruent.

right angle An angle that measures 90°.

right prism A prism in which all the lateral faces are rectangles.

right triangle A triangle with one right angle.

rigid motion A rigid motion (or transformation) is an act of moving a geometric figure from some starting position to some ending position without altering its shape or size. In other words, a rigid motion is an isometric motion.

ring A ring is an algebraic structure consisting of a set together with two binary operations (addition and multiplication). In other words, the ring is a such algebraic structure which generalize the main properties of the addition and multiplication of integers, real and complex numbers. For example, one of the most common example of a ring is the set of integers endowed with its basic arithmetic operations of addition and multiplication. This ring is actually a commutative ring, since addition and multiplication are commutative in this set.

rise The change in the second coordinate (\( y \)– coordinate) between two points on a straight line or the vertical distance between two points in a coordinate plane.
**Rolle’s Theorem** If a function \( y = f(x) \) is continuous on the closed interval \([a, b]\), differentiable on the open interval \((a, b)\) and \( f(a) = f(b) \), then there is a point \( c \in (a, b) \) such that \( f'(c) = 0 \).

Geometrically speaking, Rolle’s Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line (or x-axis).

*Michel Rolle* (1652 – 1719) was self-educated French mathematician.

**Roman Numerals** The ancient Roman used a number system that is still in use and is called the Roman system of numeration. In their latest form, the Roman numerals looked like this:

\[
\begin{align*}
I &= 1, & II &= 2, & III &= 3, & IV &= 4, & V &= 5, & VI &= 6, & X &= 10, & L &= 50, & C &= 100, & D &= 500, & M &= 1000, & CM &= 900, & MC &= 1100, & ...
\end{align*}
\]

We can see, if larger numeral precedes a smaller one, they are added (VI = V + I = 5 + 1 = 6), if the smaller one comes first (in which case the symbol is not repeated), then it is subtracted from the larger numeral (XL = L – X = 50 – 10 = 40).

**roots/solutions of an equation** The \( x \)-values for which an equation such as \( f(x) = 0 \) is true.

**root of multiplicity two** A repeated root that appears twice.

**rose curve** A rose curve is a member of a family of curves that resemble flowers. It is a graph of a polar equation of the form \( r = a \sin n\theta \) or \( r = a \cos n\theta \).

**roster notation/form** A most common way (with set-builder notation) of naming sets by listing all the elements in the set between two braces.

**rotation** A rotation is a rigid motion performed by rotating a geometric figure in the plane about specific point, called the center of rotation. The angle through which the figure is rotated is called the angle of rotation.

**rounding decimals** The procedure for rounding decimals is the same as for rounding whole numbers, (see rules below) except that we don’t add zeros at the end.

**rounding whole numbers** Using only a required number of digits and replacing the rest by zeros.

**rows of a matrix** The horizontal elements of a matrix.

**ruler** A tool for measuring length.

**rules for rounding whole numbers** 1. Locate the digit in that place. 2. Consider the next digit to the right. 3. If the digit to the right of the rounded digit is less than 5, leave the digit the same and replace the digits to the right by zeros. 4. If the digit to the right of the rounded digit is 5 or more, increase (round up) the rounded digit by one and replace the digits to the right by zeros.

**Example 1** Round 7685 to the nearest hundred.

**Solution**
1. Locate the digit in the hundreds place,  
   2. Consider the next digit to the right,  
   3. Since that digit, 8, is 5 or higher, round 6 hundreds up to 7 hundreds.  
   4. Change all digits to the right of hundreds to zeros.  

   7700 ← This is the answer.

**Example 2** Round 23.96845 to the nearest hundredth.

**Solution**

1. Find the place to which the rounding is being done (hundredths place) and draw a “cut-off” line after the hundredths place to show that you are cutting off and dropping the rest of the digits.

   23.96845  
   (You are cutting off 8, 4 and 5, and they will be dropped.)

2. Look only at the first digit (8) you are cutting off. Ignore the other digits (4, 5) you are cutting off.

3. Round up the hundredths place because 8 ≥ 5. So, 23.96845 rounded to the nearest hundredth is 23.97 or you can write 23.96845 ≈ 23.97.

**run** The change in the first coordinate (x-coordinate) between two points on a straight line or the horizontal distance between two points in a coordinate plane.

**s**

**saddle points** The critical points of a function of two variables \(f(x, y)\) are those points of which both partial derivatives are zero \((f_x = 0, f_y = 0)\). A critical point of a function of a single variable is either a local minimum, a local maximum, or neither. With functions of two variables there is a fourth possibility — a saddle point.

**Definition** A point is a saddle point of a function of two variables if

\[
f_x = 0, \quad f_y = 0, \quad \text{and} \quad f_{xx}^2 \cdot f_{yy}^2 - f_{xy}^2 < 0
\]

**sale price** The price of an item after a discount has been deducted.

**sales tax** A tax (in percent) added to the purchase price of an item.

**salvage (scrap) value** The value of an item at the end of its useful life.

**sample** A subset of a population.

**sample space** The set of all possible outcomes or events of an experiment that cannot be further broken down.

**sandwich theorem** Suppose that \(g(x) \leq f(x) \leq h(x)\) for all \(x\) in some open interval \((a, b)\) containing \(c\), except possibly at \(x = c\) itself. Suppose also that

\[
\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.
\]

Then \(\lim_{x \to c} f(x) = L\).

The functions \(g(x)\) and \(h(x)\) are said to be lower and upper bounds (respectively) of \(f(x)\).
The sandwich theorem (known also as the **squeeze theorem**, the **pinching theorem** and the **sandwich rule**) is a theorem regarding the limit of a function. This theorem enables us to calculate a variety of limits. It is typically used to confirm the limit of a function by comparison with known limit(s) of two other functions, because values of the function \( f \) are sandwiched between these two other functions.

**Example** If \( \sqrt{7 - 3x^2} \leq f(x) \leq \sqrt{7 - 2x^2} \) for \(-1 \leq x \leq 1\), find \( \lim_{x \to 0} f(x) \).

**Solution**
\[
\lim_{x \to 0} \sqrt{7 - 3x^2} = \sqrt{7 - 3 \cdot 0} = \sqrt{7} \quad \text{and} \quad \lim_{x \to 0} \sqrt{7 - 2x^2} = \sqrt{7}; \quad \text{by the sandwich theorem,}
\]
\[
\lim_{x \to 0} f(x) = \sqrt{7}
\]

**scalar** A physical quantity that involves only a magnitude and can be represented by a real number. Examples of scalars are mass, density, energy, etc.

**Scalar and Vector Fields**
Scalar and vector quantities can be functions of the coordinates of points in space and time. In this case, they are called **scalar** and **vector fields**, respectively. Generally, a **vector field** on a domain in the plane or in space is a function that assigns a vector to each point in the domain.

A field of three-dimensional vectors might have an equation like
\[
F(x, y, z) = M(x, y, z) \mathbf{i} + N(x, y, z) \mathbf{j} + P(x, y, z) \mathbf{k}.
\]

**Scalar Product (Dot Product, Inner Product) of Two Vectors** The scalar product of two nonzero vectors is a number (scalar) equal to the product of the lengths of these vectors by cosine of the angle between them.

The scalar product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is denoted by \( \mathbf{a} \cdot \mathbf{b} \). Thus by definition
\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,
\]
where \( \theta \) is the angle between the vectors \( \mathbf{a} \) and \( \mathbf{b} \).

**N. B.** If vectors \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal (perpendicular), then \( \mathbf{a} \cdot \mathbf{b} = 0 \), because \( \cos 90^\circ = 0 \).

**scalene triangle** A triangle with no congruent sides (or all sides are of different lengths).

**scatter plot/diagram** A graph/diagram that shows the relationship between two variables. Each variable appears on one of the coordinate axes, and data are plotted as points. The two coordinates of a point represent the measures of two attributes of each item. Most often a scatter diagram is used as a tool to prove or disprove cause-and-effect relationships between two variables.

**scientific* notation** A positive number written in the form \( M \cdot 10^n \), where \( n \) is an integer, \( 1 \leq M < 10 \), and \( M \) is expressed in decimal notation. A positive exponent indicates a large number and a negative exponent indicates a small number (between 0 and 1). In the other words, the scientific notation is an exponential way of writing numbers that accommodates values too large or small to be conveniently written in standard decimal notation. For example, excluding the Sun, the closest star to our planet is Proxima Centauri, which is 4.3 light-years away or 40,707,240,000,000,000m. In scientific notation this number with 17 digits can be expressed as \( 4.070724 \cdot 10^{16} \) m. Or, another case of so small number. To express mass of an electron we need 30 decimal places, but using scientific notation we will have: \( m_e = 9.1 \cdot 10^{-28} \) g (or 0.0000000000000000000000000091g).

----------
* Marie Curie (1867 – 1934), Polish/French physicist and chemist: “I am among those who think that science has great beauty.”

secant In a right triangle, the ratio of the length of the hypotenuse to the length of the side adjacent to an acute angle.

**Second Derivative Test (the second sufficient condition for an extremum)** Let \( f''(x) \) exist on some open interval \((a, b)\) containing \( x_0 \) and let \( f'(x_0) = 0 \).

1. If \( f''(x_0) > 0 \), then \( f(x_0) \) is a relative minimum.
2. If \( f''(x_0) < 0 \), then \( f(x_0) \) is a relative maximum.
3. If \( f''(x_0) = 0 \) or \( f''(x_0) \) does not exist, then this test gives no information about extrema, so we will use the first derivative test to identify relative/local minimum or maximum at the point \([x_0, f(x_0)]\).

**Example** Find any critical numbers (stationary points) and then use the second derivative test to decide whether the critical numbers (stationary points) lead to relative (local) maxima or relative (local) minima of the function \( f(x) = x^3 - 2x^2 + x + 3 \).

**Solution** First we will find the points where the first derivative is 0. Here \( f'(x) = 3x^2 - 4x + 1 \). We will solve equation \( f'(x) = 0 \) to find stationary points (critical numbers).

\[ 3x^2 - 4x + 1 = 0 \rightarrow (x - 1)(3x - 1) = 0 \rightarrow x - 1 = 0 \text{ or } 3x - 1 = 0 \rightarrow x_1 = 1 \text{ or } x_2 = \frac{1}{3}. \]

Now we will use the second derivative test.

The second derivative is \( f''(x) = 6x - 4 \).

Just now we will evaluate \( f''(x) \) at the first stationary point \( x_1 = 1 \): \( f''(1) = 6(1) - 4 = 2 > 0 \), so that by Part 1 of the second derivative test, 1 leads to a relative (local) minimum of \( f(x) \)

\( f_{min}(1) = 1 - 2(1) + 1 + 3 = 3. \)

Also, when \( x_2 = \frac{1}{3} \), \( f''\left(\frac{1}{3}\right) = 6\left(\frac{1}{3}\right) - 4 = 2 - 4 = -2 < 0 \), with \( \frac{1}{3} \) leading to a relative maximum of \( f(x) \)

\( f_{max}\left(\frac{1}{3}\right) = \frac{1}{27} - 2\left(\frac{1}{3}\right)^{2} + \frac{1}{3} + 3 = \ldots = \frac{85}{27}. \)

**second-order curve** A plane curve whose rectangular Cartesian coordinates satisfy an algebraic equation of the second degree

\[ ax^2 + bxy + cy^2 + dx + cy + f = 0. \]

**sector of a circle** A region of a circle bounded by two radii and their intercepted arc.

**semicircle** A semicircle is two-dimensional geometric shape that forms half of a circle.

**Separable Differential Equation** A separable differential equation is any differential equation that we can write in the following form

\[ \frac{dy}{dx} = G(x)F(y) \quad \text{or} \quad \frac{dy}{dx} = \frac{G(x)}{F(y)}. \]

This is the type of nonlinear first order differential equation, and for this type of equation, all \( x \) terms can be collected with its differential \( dx \) and all \( y \) terms with \( dy \), and a solution we can obtain by integration both sides. After separation of variables we will have

\[ F(y)dy = G(x)dx /\int (\text{integration}) \Rightarrow \int F(y)dy = \int G(x)dx. \]
After integration both sides we will have a general solution in implicit form.

**Example 1** Solve the equation \( \frac{dy}{dx} = \frac{3x^2 + 2}{5y} \).

**Solution** We will separate variables, and integrate:

\[
5y \, dy = (3x^2 + 2) \, dx \quad \Rightarrow \quad \int 5y \, dy = \int (3x^2 + 2) \, dx \quad \Rightarrow \quad 5 \int y \, dy = 3 \int x^2 \, dx + 2 \int dx \quad \Rightarrow \\
\frac{5y^2}{2} = \frac{3x^3}{3} + 2x + C_1 \quad \Rightarrow \quad y^2 = \frac{2}{5}x^3 + \frac{4}{5}x + C, \text{ where } C = \frac{2}{5}C_1.
\]

**Example 2** Solve the equation \( y' = \frac{xy \sin x}{1+y} \).

**Solution** The given equation we will write in the form \( \frac{dy}{dx} = \frac{xy \sin x}{1+y} \).

We separate the variables:

\((1 + \frac{1}{y})dy = x \sin x \, dx.\)

Integrating the left-hand side of this equation with respect to \( y \):

\[
\int (1 + \frac{1}{y}) \, dy = \int dy + \int \frac{dy}{y} = y + \ln |y| + C_1,
\]

and the right-hand side with the respect to \( x \) (Using Integration by Parts: \( \int u \, dv = uv - \int v \, du \)):

\[
\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C_2, \quad \Rightarrow \text{ let } u = x \Rightarrow du = dx, dv = \sin x \, dx \Rightarrow v = \int \sin x \, dx = -\cos x + C_2
\]

\( y + \ln |y| = \sin x - x \cos x + C, \quad (1) \)

where \( C = C_2 - C_1 \) is an arbitrary constant of integration. This is the general solution in the implicit form.

To find the general solution of the given differential equation in the explicit form \( y = f(x) \), we have to solve equation (1) with respect to \( y \). Unfortunately, this is impossible to be done, since the solution cannot be expressed in terms of elementary functions.

**sequence** A list/set of numbers that follows a certain pattern/order.

**series** The sum of the specified terms in a sequence.

* Famous Indian mathematician and astronomer Brahmagupta (598 – 668 AD) provided so elegant results for the summation of series of squares and cubes:

\[
1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6};
\]

\[
1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2 = \left[ \frac{n(n+1)}{2} \right]^2
\]

**set** A collection of objects united by some common feature.

**set-builder notation** The naming the set by describing basic characteristics of the elements (members) in the set, without listing them. It is written between two braces. For example, \( \{x | x \leq 4 \} \), which is read “The set of all \( x \) such that \( x \) is less than or equal to 4.”
sexagesimal system of numbers The ancient Babylonians, approximately 6000 years ago, developed a positional number system, and the number 60 was used as a base, as we use 10 in our number system.

sigma notation The naming of sum using the Greek capital letter ∑ (sigma) as a part of an abbreviated form; also called summation notation. For example, $\sum_{k=1}^{5} k^2$ means sum of squares of the first five natural numbers, i.e. $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$.

SI Prefixes The SI (International System of Units*) prefixes or metric prefixes used to form the names and symbols of decimal multiples and submultiples of SI units are given in the attached table. The following 20 SI prefixes can be used to prefix any of the units to produce a multiples or submultiples of the original unit.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{24}$</td>
<td>yotta</td>
<td>Y</td>
</tr>
<tr>
<td>$10^{21}$</td>
<td>zetta</td>
<td>Z</td>
</tr>
<tr>
<td>$10^{18}$</td>
<td>exa</td>
<td>E</td>
</tr>
<tr>
<td>$10^{15}$</td>
<td>peta</td>
<td>P</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>tera</td>
<td>T</td>
</tr>
<tr>
<td>$10^{9}$</td>
<td>giga</td>
<td>G</td>
</tr>
<tr>
<td>$10^{6}$</td>
<td>mega</td>
<td>M</td>
</tr>
<tr>
<td>$10^{3}$</td>
<td>kilo</td>
<td>k</td>
</tr>
<tr>
<td>$10^{2}$</td>
<td>hecto</td>
<td>h</td>
</tr>
<tr>
<td>10</td>
<td>deka</td>
<td>da</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>deci</td>
<td>d</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>centi</td>
<td>c</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>milli</td>
<td>m</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>micro</td>
<td>μ</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>nano</td>
<td>n</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>pico</td>
<td>p</td>
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<td>z</td>
</tr>
<tr>
<td>$10^{-24}$</td>
<td>yocto</td>
<td>y</td>
</tr>
</tbody>
</table>

*There are seven fundamental units from which all others are derived and they are dimensionally independent. International System of Units (SI) is actually derived from this seven basic units to provide for measurement of mass (kilogram: kg), time (second: s), length (meter: m), electrical current (ampere: A), thermodynamic temperature (kelvin: K), quantity of matter (mole: mol) and luminous intensity (candela: cd).
similar polygons  The polygons in which corresponding sides are proportional and corresponding angles are congruent (the same measure).

simple harmonic motion  An oscillatory motion about an equilibrium position. If friction is neglected, this specific motion is periodic and repeats indefinitely and can be described by a sinusoid or other trigonometric graphs. For example, the position of a point oscillating about an equilibrium position at time \( t \) is modeled by either

\[
s(t) = a \cos \omega t \quad \text{or} \quad s(t) = a \sin \omega t,
\]

where \( a \) and \( \omega \) are constants, with \( \omega > 0 \). The amplitude of the motion is \( |a| \), the period is \( \frac{2\pi}{\omega} \), and the frequency is \( \frac{\omega}{2\pi} \) oscillations per time unit.

simple interest  In simple interest, interest is paid only on the principal, using formula \( I = P \cdot r \cdot t \), where \( P \) is the principal (in dollars), \( r \) is the interest rate (%), and \( t \) is time (in years).

simple random sample  A sample selected so that each item or person in the population has the same chance of being included.

simplex method  The simplex method is popular algorithm for solving linear programming problems. This method starts with the selection of one arbitrary corner point from the feasible region, and after certain number of steps an optimum solution will be reached.

simplifying an expression  The process of finding the simplest form of an expression.

Simpson’s* rule  Let \( f(x) \) be continuous function on \([a, b]\) and let \([a, b]\) be divided into an even number \( n \) of equal subintervals by the points \( a = x_0, x_1, x_2, \ldots, x_n = b \). Then by Simpson’s rule,

\[
\int_a^b f(x) \, dx \approx \left( \frac{b-a}{3n} \right) \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right].
\]

………………

*Thomas Simpson (1710-1761), British mathematician.

simulation  A procedure for determining the probability of real events by running experiments that closely resemble the real situation.

sine  In a right triangle, the ratio of the length of the side opposite to an acute angle to the length of the hypotenuse.

sine wave (sinusoid)  The graph of a sine function is called a sine wave or sinusoid (or a sine curve).

sinking funds  A sinking fund* is a special type of annuity in which the goal is to save a specific amount of money in a specific amount of time.

…………………………

* Sinking Fund Payment Formula
\[ p = \frac{A\left(\frac{r}{n}\right)}{(1 + \frac{r}{n})^{nt} - 1}, \]

where \( p \) is the payment needed to reach the accumulated amount \( A \). Payments are made \( n \) times per year, for \( t \) years, into a sinking fund with interest rate \( r \) as a decimal number, compounded \( n \) times per year.

**Example**

**Saving for a New Car** James Miller would like to have $25,000 to buy a new car in 5 years. To accumulate $25,000 in 5 years, how much should he invest monthly in a sinking fund with 9% interest compounded monthly?

**Solution** Since we are looking for how much James Miller should invest each month in 5 years to obtain an accumulated amount, we will use the sinking fund payment formula. The accumulated amount, \( A \), is $25,000; the interest rate, \( r \), is 9%, or 0.09; the payments are made monthly and the interest is compounded monthly, so \( n \) is 12, and the number of years, \( t \), is 5.

\[
P = \frac{A\left(\frac{r}{n}\right)}{(1 + \frac{r}{n})^{nt} - 1} = \frac{25,000\cdot(0.09)}{(1 + \frac{0.09}{12})^{60} - 1} = \frac{25,000 \cdot 0.0075}{1.0075^{60} - 1} = \frac{187.5}{1.565681027-1} = \frac{187.5}{0.565681027} \approx 331.4588806
\]

We often round to the nearest cent, or in this case, to $331.46.

**skew lines** Straight lines that are not coplanar and do not intersect.

**slope** The slope \( m \) of a straight line that contains the points \((x_1, y_1)\) and \((x_2, y_2)\) is the ratio \( m = \frac{y_2 - y_1}{x_2 - x_1} \) (vertical change* to horizontal change or rise to run). In other words, the slope is the measure of steepness (or slant) of a straight line, and it is a single number that allows us to determine the direction in which a line is slanting from left to right, as well as how much slant there is to the line.

*Heraclitus (544 BC – 483 BC): "Change alone is unchanging."

**slope field** A slope field (or direction field) is a graphical representation of the solutions of a first-order differential equations because the most majority of a first-order differential equations cannot be solved. This graphical method gives a visualization of the general shape of the solution curves, and it is achieved without solving differential equation analytically.

**slug** In the English system, where weight is measured in pounds, mass is measured in slugs. Thus, \( \text{pounds} = \text{slugs} \cdot 32 \), assuming the gravitational constant is \( 32 \frac{\text{ft}}{\text{sec}^2} \) (N.B. 1 slug = 14.59 kg).

**Socratic* teaching** Socratic teaching is the oldest, and still the most powerful, teaching tactic for fostering critical thinking. In Socratic teaching we focus on giving students questions, not answers (Paul, R and Elder, *Foundation For Critical Thinking*).

*Socrates (470/469 BC – 399 BC), famous Greek philosopher and one of the founders of Western philosophy.
solid A solid is any bounded set of points in space.

solid of revolution An object formed by rotating a plane figure about an axis in space.

solution/root A number that makes the equation a true statement. In other words, it is a value, such that, when you replace the variable with it, it makes the equation true (the left side comes out equal to the right side).

solution set The set off all numbers that satisfy the equation. Those objects are generally called elements (members) of the set.

solving triangles To solve a triangle means to find the measures of all the angles and sides of triangle. In other words, the solution of triangles is finding unknown quantities of a triangle from given values of other of its quantities.

sphere* The set of points in space that are equidistant from a given point known as the center of the sphere.

.................
* Oswald Spengler, German philosopher of history (1880 – 1936): “What is outside the largest sphere?”

square A parallelogram with all sides and angles congruent.

square matrix A matrix with the same number of rows and columns.

square root If \( b^2 = a \), then \( b \) is square root* or radical of \( a \), i.e. \( b = \sqrt{a} \). The symbol \( \sqrt{ \) is called radical sign, and the number/expression inside the radical sign \( a \) is called radicand. The radical symbol was first used by Leonardo Pisano Fibonacci (c. 1170 – c. 1250), famous Italian mathematician, and, for sure, the greatest European mathematician of the middle ages. He was especially famous for his contributions to number theory. Hi used first time the square root notation c. 1202 in his famous book *Liber abaci/The Book of calculating.*

Generally speaking, if \( b^n = a \), then \( b \) is \( n \)th root of \( a \), i.e. \( b = \sqrt[n]{a} \).

.................
* 1) The Product Rule: For any nonnegative radicands \( A \) and \( B \), \( \sqrt{A} \cdot \sqrt{B} = \sqrt{A \cdot B} \);

2) The Quotient Rule: For any nonnegative radicand \( A \) and any positive radicand \( B \), \( \frac{\sqrt{A}}{\sqrt{B}} = \sqrt[ ]{\frac{A}{B}} \).

Squaring the Circle Squaring the circle is one of the famous three great problems of Classical Geometry, along with the *trisection of the angle* and the *duplication of the cube*. It is impossible to construct a square equal in area to a given circle with compass and straightedge.
standard deviation \( s \) (or \( \sigma \)) A measure of dispersion which measures how much the data differ from the mean. Formula for the standard deviation is given by \( s \) or \( \sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} \), where \( n \) is the number of pieces of data, \( x_i \) is the individual data, and \( \bar{x} \) is the mean.

standard form of a linear equation An equation of the form \( Ax + By = C \), where \( A \), \( B \), and \( C \) are real numbers and \( A \) and \( B \) are not both 0.

standard score* Also called \( z \) – score, it is the number of standard deviations that a given value is above or below the mean. In other words, standard score (aka \( z \)-score) indicates how many standard deviations an element is from the population mean.

\( * \)Standard Score (z-score) Formula: \( z = \frac{\text{value of the piece of data} - \text{mean}}{\text{standard deviation}} = \frac{x - \mu}{\sigma} \)

standard unit vectors The standard unit vectors in the three-dimensional coordinate system \( XYZ \) are \( i = \langle 1, 0, 0 \rangle \), \( j = \langle 0, 1, 0 \rangle \), and \( k = \langle 0, 0, 1 \rangle \).

statement A declarative sentence that is either true or false, but not both simultaneously.

stationary point A stationary point is a point \( x_0 \) at which the first derivative of a function \( f(x) \) vanishes: \( f'(x_0) = 0 \). A stationary point may be a minimum, maximum or inflection point.

stationary value See “critical point”.

statistic A number that describes a set of data or measured characteristic of a sample.

statistics One of over 3000 branches of Mathematics that deals with collection, organization, display, analysis and interpretation of the numerical data.

Stem-and-Leaf Displays A method of organizing data. This powerful method in statistics can provide an ‘at a glance’ tool for specific information in large sets of data. To construct a stem-and-leaf display each value is represented with two different groups of digits. The left group of digits is called stem. The remaining group of digits on the right is called the leaf. For example, 54|7 represents 547.

stochastic process/random process A collection of random variables. In other words, this is a process determined by a random distribution of probabilities, referring to a behavior not governed by known equations and initial condition, thus unpredictable at any past or future time.

stochastics* A branch of applied mathematics concerned with the collection and interpretation of quantitative data and the use of probability theory to estimate population parameters.

\( * \)Stochastic is the term for random or uncertain, determined by chance. Deterministic is the opposite of stochastic.
**straight angle**  An angle measuring exactly $180^\circ$.

**stratified sampling**  Stratified sampling in statistics is a sampling technique where the entire population is divided into parts, called strata, for the purpose of drawing a sample. This procedure involves dividing the population by characteristics called *stratifying factors* such as gender, race, occupation, religion, or age. The use of this sampling technique requires some knowledge of the population.

**subset**  A set that is the part of other (larger) set.

**substitute**  To replace a variable with a number or expression.

**Substitution Method**  An algebraic method for solving systems of equations. This method, if we have system of two equations, consists in the following:

1. **using one equation** (you will see which one is easier) solve for one of the unknowns, say $x$, in terms of the known quantities and the other unknown, $y$;
2. **substitute this expression** into the other equation, which then contains only one unknown, $y$;
3. **solve the equation** and find the value of $y$;
4. **substitute value of $y$** into the expression of the unknown $x$ which was found at the beginning of the solution. This yields the value of $x$.

**Example 1**  Solve the system

\[
\begin{align*}
x + 2y & = 10, \quad (1) \\
3x + 4y & = 8. \quad (2)
\end{align*}
\]

**Solution**  We solve one equation for one variable. Since the coefficient of $x$ is 1 in equation (1), it is pretty easier to solve that equation for $x$:

\[
x + 2y = 10 \quad \rightarrow \quad x = 10 - 2y \quad (3)
\]

We can see that $x$ and $10 - 2y$ name the same thing. Thus in equation (2), we can substitute $10 - 2y$ for $x$:  

\[
3x + 4y = 8 \quad (2) \rightarrow \quad 3(10 - 2y) + 4y = 8 \quad \text{(Substituting $10 - 2y$ for $x$)}
\]

The last equation has only one variable, $y$. We will solve it:

\[
\begin{align*}
30 - 6y + 4y & = 8 \quad \text{(Removing parentheses)} \\
30 - 2y & = 8 \quad \text{(Collecting like terms)} \\
-2y & = -22 \quad \text{(Simplifying)} \\
y & = \frac{-22}{-2} \\
y & = 11.
\end{align*}
\]

We have found the $y$-value of the solution. To find the $x$-value, we will go back to either to the original equations (1) or (2) or to equation (3), which we solved for $x$. We can see that is easier to use the equation (3). We substitute 11 for $y$ in equation (3) and compute $x$:  

\[
x = 10 - 2y = 10 - 2(11) = 10 - 22 = -12.
\]

The ordered pair (-12, 10) may be a solution. We will check as follows.

\[
\begin{align*}
\text{Check:} \quad x + 2y & = 10 \\
-12 + 22 & = 10 \quad (?) \\
10 & = 10 \quad (\text{True})
\end{align*}
\]

\[
\begin{align*}
3x + 4y & = 8 \\
3(-12) + 4(11) & = 8 \quad (?) \\
-36 + 44 & = 8 \quad (?) \rightarrow 8 = 8 \quad (\text{True})
\end{align*}
\]
Since \((-12, 10)\) checks, it is the solution. We could also express the answer as \(x = -12, y = 11\) or the solution set is \{(-12, 10)\}.

**Example 2**  
John would like to make 12 gallons of 24% green pigment paint. Using a 16.5% green pigment paint and 28% green pigment paint, how many gallons of each should John use?

**Solution**

**a)** Familiarize. In this mixture problem John uses \(x\) gallons of the first green pigment and \(y\) gallons of the second.  
**b)** Translate. Since the total is to be 12 gal, we have:
- Total amount of pigment paint: \(x + y = 12\)
- Strength of the first pigment is 16.5% (0.165), strength of the second is 28% (0.28) and of mixture is 24% (0.24).

That means \(0.165x + 0.28y = (0.24)(12)\). or \(0.165x + 0.28y = 2.88\)

Just now we have a system of two equations:

\[
\begin{align*}
x + y &= 12 \\
0.165x + 0.28y &= 2.88 \\
\text{(We clear decimals in second equation by multiplying by 1000)}
\end{align*}
\]

**c)** Solve. We will use the substitution method, after simplifying the second equation.

\[
\begin{align*}
x + y &= 12 \\
165x + 280y &= 2880 \div 5 \quad \text{(simplifying)} \\
x &= 12 - y \quad \text{(Substitution method)} \\
33x + 56y &= 576 \\
33(12 - y) + 56y &= 576 \quad \text{(distributive property)} \\
396 - 33y + 56y &= 576 \\
23y &= 180, \text{ and } y = \frac{180}{23} \text{ or } y = 7.83 \text{ gal} \\
x &= 12 - 7.83 = 4.17 \text{ gal}
\end{align*}
\]

**d)** Check. The sum of 4.17 and 7.83 is 12. Also, 16.5% of 4.17 is 0.688, and 28% of 7.83 is 2.1924. These add up to 2.88.

**e)** State. John use 4.17 gal of the 16.5% green pigment and 7.83 gal of the 28% green pigment.

**Subtracting Real Numbers**  
To subtract, add the opposite (or additive inverse) of the number being subtracted (subtrahend): \(a - b = a + (-b)\) or \(a - (-b) = a + b\). In other words, to subtract \(b\) (subtrahend), you will add **opposite** of \(b\).

**subtraction** The reverse of addition.

**subtrahend** In subtraction, the number/expression being subtracted. For example, in case \(12 - 7 = 5\), the number 12 is the **minuend**, 7 is the **subtrahend**, and 5 is the **difference (remainder)**.

**success** The desired outcome in the probability theory is called **success**.
**sufficient condition**  In conditional statement “If A, then B”, the antecedent A refers to a sufficient condition for the consequent B.

**sufficient condition for inflection**  If the second derivative of the function \( y = f(x) \) exists in a neighborhood of the point \( c \), with \( f''(c) = 0 \), and the sign of \( f''(x) \) are different on the left and on the right of the point \( c \), then the graph of the function has an inflection at the point \( [c, f(c)] \). In determining intervals where a function is concave upward or concave downward, we first find domain values (x-values) where \( f''(x) = 0 \) or \( f''(x) \) does not exist. Then we test all intervals around these values in the second derivative of the function. If \( f''(x) \) changes signs around \( x_0 \), then \( [x_0, f(x_0)] \) is a point of inflection of the function. The second derivative of a function may also be used to determine the general shape of its graph on selected interval. A function \( f(x) \) is said to be **concave upward** on an interval if \( f''(x) > 0 \) at each point in the interval and **concave downward** on an interval if \( f''(x) < 0 \) at each point in the interval.

**sum/total** The number/expression obtained by addition.

**sum of the probabilities**  The sum of the probabilities of all possible outcomes of an experiment is 1.

**sum of two cubes**  \( a^3 + b^3 \)  This algebraic expression can be expressed in the factored form as the special product i.e. \( a^3 + b^3 = (a + b)(a^2 - ab + b^2) \).

**SUM OF THE TERMS OF AN INFINITE GEOMETRIC SEQUENCE**  See “convergent infinite geometric sequence”.

**supply function**  \( S(q) \)  The function of \( q \), the quantity produced (almost the linear function). In other words, a function modeling the relationship between the price \( p \) of a good and the quantity \( q \) of that good supplied.

**supplementary angles/supplements**  Two positive angles are supplementary angles (or supplements) if the sum of their measures is 180°.

**surd**  (See irrational number)  If we can’t simplify a number to remove a square root (or 3rd root, etc.) then it is a surd. In fact, surd use to be another name for irrational number.

**surface area**  The sum of the areas of all of the faces of three-dimensional figure.

**surjection**  A function \( f \) from a set \( A \) to a set \( B \) (\( f: A \rightarrow B \)) is surjective (or onto), or a surjection, if every element \( y \) in \( B \) (\( y \in B \)) has corresponding element \( x \) in \( A \) (\( x \in A \)) so that \( f(x) = y \). Multiple elements of \( A \) might be turned into same element of \( B \) by applying surjection \( f \). Definition of a surjective function \( f \) we can express symbolically, let \( f: A \rightarrow B \), then \( f \) is said to be surjection if \( \forall y \in B, \exists x \in A, f(x) = y \).

**symbol**  (See list of symbols)  A pattern or image used instead of words. Symbols are used in mathematical notation to express a formula or to replace a constant.
symmetry* The symmetry is an equivalence relation. Two figures/objects are symmetric to each other with respect to the invariant transformations if one figure/object is obtained from the other by one of the invariant transformations. In the broad sense of the word, symmetry is to be understood as any regularity in the inner structure of a body or figure. Geometrically speaking, a symmetry of a geometric figure is rigid motion that moves the figure back onto itself. That is, the beginning position and ending position of the figure must be identical.

*The word **symmetry** comes from the Greek meaning balanced proportions.

**Symmetric Property of Equality** For all real numbers \( a \) and \( b \), if \( a = b \), then \( b = a \).

**Symmetry with respect to the coordinate axis** (1) The graph of the some relation is **symmetric with respect to the x-axis** if the replacement of \( y \) with \( -y \) results in an equivalent equation of that relation. (2) The graph of the some relation is **symmetric with respect to the y-axis** if the replacement of \( x \) with \( -x \) results in an equivalent equation of that relation.

**Symmetry with respect to the origin** The graph of the some relation is **symmetric with respect to the origin** if the replacement of both \( x \) with \( -x \) and \( y \) with \( -y \) results in an equivalent equation of that relation.

**Synthetic division** Synthetic division is a shortcut method of dividing a polynomial
\[
a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0
\]
by a binomial of the form \( x - k \), using only \( k \) and polynomial coefficients \( a_n, a_{n-1}, \ldots, a_2, a_1, \) and \( a_0 \).

**System of equations** A set of two or more equations, in two or more variables that are considered at the same time and to be solved simultaneously. **N. B.** A solution of a system of equations must satisfy every equation in the system.

**System of inequalities** A set of two or more inequalities that are considered at the same time.

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**T**

table A method of presenting data in rows and columns.

**Table of simplest integrals (tabular/standard integrals)**

<table>
<thead>
<tr>
<th>1) ( \int x^n , dx = \frac{x^{n+1}}{n+1} + C ), ( n \neq -1 ), where ( C ) is an arbitrary constant.</th>
<th>2) ( \int a^x , dx = \frac{a^x}{\ln a} + C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3) ( \int e^x , dx = e^x + C )</td>
<td>4) ( \int a^{kx} , dx = \frac{a^{kx}}{k \ln a} + C )</td>
</tr>
</tbody>
</table>
tally  The use of marks for counting.

tangent  In a right triangle, the ratio of the length of the side opposite an acute angle to the length of
            the side adjacent to it.

tangent to a curve  A straight line in the plane of a curve that intersects a curve at exactly one point.

Tartaglia’s Formula  Italian mathematician Niccolo Tartaglia (1500-1557) developed method of solving a
cubic equation of the form \( x^3 + mx = n \). His formula/rule (published 1545) for finding the one real
solution of the equation is

\[
x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.
\]

tautology  A statement that is always true. In the formal logic, tautology is universal unconditional
            truth, always valid.

Taylor Series*  A Taylor Series is a series expansion of a function \( f(x) \) about a point \( x = x_0 \) and it is given by

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.
\]

In other words, the Taylor Series is an representation of a function as an infinite sum of terms. If \( x_0 = 0 \),
the expansion is known as a MacLaurin Series and it is given by

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.
\]
1. \[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \text{ (Interval of Convergence)} \]

2. \[ \frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1 \]

3. \[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty \]

4. \[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \]

5. \[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty \]

6. \[ \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1 \]

7. \[ \ln \left( \frac{1+x}{1-x} \right) = 2 \tan^{-1} x = 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n+1}}{2n+1} + \ldots) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1 \]

8. \[ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1 \]

9. \[ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \]

10. \[ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad |x| < \infty \]

**tensor** Tensors are simply mathematical objects that can be used to describe physical properties just like scalars and vectors. In fact tensors are merely a generalization of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor. The rank (or order) of a tensor is defined by the number of directions required to describe it.

**term** A number, a variable (letter), or a product of a number and one or more variables (3x, -7abc, 11x^2, 2y, ...). In other words, the terms of an algebraic expression are those parts that are separated by *addition*. For example, the algebraic expression 5x^3 + 2x^2 - 7x + 9 has four terms (polynomial of 3\(^{rd}\) degree): 5x^3, 2x^2, -7x, and 9. Note that -7x is a term, rather than 3x because subtraction is actually a form of addition. Namely, 

\[ 5x^3 + 2x^2 - 7x + 9 = 5x^3 + 2x^2 + (-7x) + 9 \]

**terminal point** When two letters A and B are used to name a vector \( \overrightarrow{AB} \), the second letter indicates the terminal (ending) point of the vector.

**terminal side** When a ray is rotated around its endpoint to form an angle, the ray in its location after rotation is called the terminal side of the angle.

**terminating decimal** A decimal that can be written using a finite number of decimal places.
**tessellation**  Tessellation (or tilling) is a pattern consisting of the repeated use of the same geometric figures to entirely cover a plane, leaving no gaps. The geometric figures used are called the tessellating shapes of the tessellation.

**tesla**  The international measure unit of the magnetic induction (in the meter-kilogram-second system, MKS).

*About Nikola Tesla, “The Serbian God of Lightning's and The Wizard of Electricity”

There are very few scientists in the world like, well known, Nikola Tesla. Today, it is, less or more, clear that he was not only an ingenious inventor, discoverer, and futurist, but also an esotericist. Such a unity of science and mysticism we can find, maybe, with Isaac Newton and Leonhard Euler, or in earlier periods with the Renaissance magi. All the Tesla's discoveries, it seems to us today, come from the ancient search for the God and devotion to the secret of the creation of the world.

He was born on July 10, 1856 as the son of the Serbian orthodox priest Milutin Tesla from the village of Smiljan, in the Lika region in the Austro-Hungarian territory. He completed high school in Gospich, and studied technical sciences in Graz and Prague. As an engineer he worked in the Telephone Enterprise in Budapest and Edison’s company in Paris and New York. After this experiences he founded his own laboratory where he worked until his death, and this cherubic work was just like - "The Divine Flash in the Darkness of Time".

Majority of the basic findings in the large field of electrical engineering belongs to this romantic dreamer of the global human prosperity. All in all, more than a thousand patents and discoveries in all fields of electrical engineering bear his name. Many of his other discoveries was usurped of others with his unbelievable nonchalance. However, a huge group of his patents in the domain of producing, transferring and using polyphase alternating currents from the period 1887-1890 represent the cornerstone of the modern electro-energetic, as we can read in the chronicles of the science development.

His patents are unavoidable in the construction of the first big generators for the polyphase currents. One plate on the generator placed in the hydroelectric plant at the Niagara Falls, dating from 1899, is a witness on Tesla's grand role in the venture which has been taken as the beginning of the electrification of the world, but he thought wider then that. For an example, the invention of the turbine without blades (!), as well as of the pump and the rotation counter based upon same principle (1913) - in which he invested 20 years of life and work - only in our time have become completely understandable and due to their originality induce excitement in the scientific world. His famous transformer and radio-connection are not only discoveries but an entirely a new chapter in the history and future of the technical sciences. Almost a century ago he managed shockingly, with the help of oscillators, to produce high frequency currents of several tens of thousands period units of more than a million voltages. His experiments in New York and Colorado Springs look more like rites of superhuman intelligence, and his Diary of Researches in Colorado Springs seems as if it was a record on devotion.

The oscillators he made are used even today in radio technique, in industry, in the process of nuclear power energy release, and Tesla’s old prediction that it would be applied in the medicine came true. The wireless production of light and the experiments with tubes filled with rarefied gas represent the great beginning of the modern technique of luminescent light. Konrad Wilhelm Roentgen wondered and was curious when he had received from Tesla images of parts of a human body created due to the Roentgen ray tubes filled with Tesla’s high frequency currents. It were these experiments
with the high frequency current, well known as Tesla's current in the science today, which established the technique of the wireless. The verdict which protected him, after the court proceedings against Guglielmo Marconi, was confirmed by the Supreme Court in USA. Tesla's demonstration of tele-directing, the wireless remote control, today is famous and made perpetuated in movies, and the reconstruction of the model of the then used vessel can be seen nowadays in the museum “Nikola Tesla” in Belgrade. His research paved the way for launching of the tele-automatic in the years to come. The antenna at the “World Radio Station”, erected on the Long Island in 1900, urged doubts and was considered the science fiction, and today it is a part of everyday life even in the most remote parts of the planet. At the same time Tesla started to retreat to the altitude from the swarm of the world. It can be considered as a result of a disappointment due to the failure to require financial support to continue his researches on the Long Island, whereas it can also be attributed to the puzzling white dove which regularly visited his window in New York. One thing is certain, from then on he lived in a sort of urban ghetto, alone in the crowd. It is known that he worked equally intensively, some findings were announced and became public accessible, but it is less and less clear what were his main scientific aims, and how far he had gone in his research. In nowadays there are many speculations about that.

Nikola Tesla (1856 – 1943)

He received for his work a great number of recognitions of the greatest world universities and scientific institutions. The international commission for electrical engineering, at the session in Philadelphia in 1960, seventeen years after his death, decided that the international measure unit of the magnetic induction (in the meter-kilogram-second system) be should named Tesla. Thus, he was ranked among the greatest researchers such as Volta, Watt, Ampere, Newton, Faraday, Kelvin... to name but a few. He was eighty years old when he wrote: "I don't need help, I need difficulties. The harder, the better. I work best when entangled in a battle!"

Nikola Tesla died on January 7th, 1943 in the Hotel New Yorker, room 3327 on the 33rd floor, where he had lived for the last ten years of his life. He was cremated and the urn with his ashes was temporary placed at one of New York cemeteries, in 1957, upon his personal request, it was transferred to Belgrade. That golden ball on the stone pedestal, that looks like a mystical
nucleus of the world or an ancient picture of the universe, is kept in his museum in Belgrade and can be seen by all visitors. All his inheritance, 150,000 of versatile testimonies, was bequeathed to his fatherland. It served as a base for establishing a unique museum, entirely dedicated to the mysterious, yet puzzling genius, the genius who lit the world.

**Tests for divisibility**

**Divisibility by** 2: Any even number.

3: Sum of digits is divisible by 3.

4: Last two digits are zeros or form a number divisible by 4.

5: Ones digit is 0 or 5.

6: Divisible by both 2 and 3.

7: Actually divide.

8: Last three digits are zeros or form a number divisible by 8.

9: Sum of digits is divisible by 9.

10: Ones digit is 0.

11: The number 11 divides only those numbers whose sum of digits occupying odd positions is either equal to the sum of digits occupying even positions or differs from it by a number that is divisible by 11. For example, The number 103,785 is divisible by 11 since the sum of the digits in odd positions (1+3+8 = 12) is equal to the sum of the digits in even positions (0+7+5 = 12). The number 9,163,627 is divisible by 11 since the sum of digits in odd positions is 9+6+6+7 = 28, while the sum of digits in even positions is 1+3+2 = 6; the difference between 28 and 6 is 22, that is divisible by 11.

12: Divisible by both 3 and 4.

25: Last two digits are zeros or form a number that is divisible by 25.

**tetrahedron** Tetrahedron is a polyhedron composed of four equilateral triangles, three of which meet at each corner or vertex.
**Thales’ Theorem** The diameter of a circle always subtends a right angle to any point on the circle. The converse of Thales’ theorem is also valid; it states that *a right triangle’s hypotenuse is a diameter of its circumcircle.*

*With Thales of Miletus (ca. 624 BC – ca. 546 BC), one of the Seven Sages of Greece, begins the western philosophy, according Bertrand Russell. He is credited, in addition to, with the first use of the deductive reasoning applied to geometry. He is also the first known individual to whom a mathematical discovery has been attributed and he was also the first person who we can free call studied electricity. Generally speaking, he was pioneer in many areas of human thought. For example, he divided year in 365 days or he was able to determine the height of a pyramid by measuring the length of its shadow at particular time of day. To Thales the primary question was not what do we know, but how do we know it. He believed water to be the primary element from which all matter was made.*

**Theorem** A statement/property that can be proved using axioms, definitions, and other already proved theorems.

**Theoretical probability** Theoretical probability is determined through a study of the possible outcomes that can occur for the given experiment. The theoretical probability of an event $E$, $P(E)$, can be determined by the following formula.

$$P(E) = \frac{\text{number of outcomes favorable to event } E}{\text{total number of outcomes}}$$

**Time** A nonspatial continuum in which events occur in apparently irreversible succession from the past through the present to the future.

*For this controversial metaphysical question of all questions Saint Augustine (354–430 A. D.) remarks that time is at once familiar and deeply mysterious. “What is time?” he asks. “If nobody asks me, I know; but if I were desirous to explain it to one that should ask me, plainly I know not.”*
Giordano Bruno (1548-1600) left us so interesting consideration about time: “Time is the father of the truth, its mother is our mind ... Time takes all and gives all.”

Famous Persian mathematician, astronomer, and writer Omar Khayyam (1048 – 1131) measured the length of the year as 365.24219858156 days. This is incredible degree of accuracy because we know that the length of the year is changing in the sixth decimal place over a person’s life time. For comparison, the length of the year today is 365.242190 days.

topology A branch of mathematics that studies the properties of spaces that are invariant under any continuous deformation.

torque The torque is the tendency of a force to produce the rotation about an axis.

torsion The torsion function $\mathbf{t}$ of a smooth space curve $C$ parametrized by arc length $s$ is given by

$$\mathbf{t} = \frac{d\mathbf{B}}{ds} \cdot \mathbf{N},$$

where $\mathbf{B}$ is unit binormal vector and $\mathbf{N}$ unit normal vector.

The torsion measures the turnaround of the binormal vector. Actually, torsion measures how the curve twists. It means that the torsion measures the failure of a curve to be a planar. The larger the torsion is, the faster rotates the binormal vector around the axis given by the unit tangent vector. In the other words, the torsion of a space curve $C$ is the rate of change of the curve’s osculating plane.

torus A three-dimensional donut-shaped surface.

total cost The money spent to produce a product.

total price The sum of the purchase price of an item and the sales tax on the item.

total profit The money taken in less the money spent, or total revenue minus total cost [$Profit = Revenue – Cost$ or $P(x) = R(x) – C(x)$].

total revenue The money taken in from the sale of product.

total value of money flow If $f(x)$ is the rate of money flow, then the total money flow over the time interval from $x=0$ to $x=t$ is given by $\int_0^t f(x)dx$.

transcendental* or non-algebraic number An irrational number which cannot be the root (solution) of any algebraic/polynomial equation with integral coefficients. For example, $\pi$ * and $e$** are transcendental numbers.

* In 1844 French mathematician Joseph Liouville (1809 – 1882) was the first to prove the existence of transcendental numbers

** Ferdinand von Lindemann (1852 – 1939), German mathematician, proved in 1882 that $\pi$ was transcendental number.
In 1882 French mathematician Charles Hermite (1822 – 1901) proved that the number e was transcendental.

transcendental function A function which cannot be given by any algebraic expression involving only their variables and constants, i.e., cannot be expressed in terms of algebra. Trigonometric, inverse trigonometric, exponential, logarithmic, and many other functions are transcendental functions.

transformation The movement of a figure in a plane from the original position, the preimage, to a new position, the image.

Transitive Property of Equality For all real numbers a, b, and c, if a = b and b = c, then a = c.

translation A translation is a rigid motion that moves a geometric figure by sliding it along a straight line segment in the plane. The direction and length of the line segment completely determine the translation.

transposition The process of moving a quantity from one side of an equation/inequality to the other side by changing its sign of operation (“grass is always greener on the other side” 😊).

transversal A straight line that intersects two or more coplanar straight lines, rays, or segments, each at different point.

transverse axis of the hyperbola The line segment that has the vertices of a hyperbola as endpoints. The length of the transverse axis is 2a.

trapezoid A quadrilateral with only one pair of parallel sides.

trial A trial is a simulation of a single event. Experimental probability is found by conducting many trials simulating the event and dividing the number of successful trials by the total number of trials.

triangle A polygon with three sides (the simplest polygon).

triangle inequality theorem The sum of the lengths of any two sides of a triangle is greater than length of the third side (or any side of a triangle is always shorter than the sum of the other two sides).

trigonometric equation An equation involving an unknown quantity under the sign of a trigonometric function is called a trigonometric equation. Equations with trigonometric functions can usually be solved using algebraic methods and trigonometric identities.

trigonometric form of a complex number The expression $z = r \cos \theta + i \sin \theta$ is called trigonometric form (or polar form) of the complex number $z = x + yi$ (standard form of a complex number). The number $r$ is the absolute value (or modulus) of complex number $z = x + yi$, and $\theta$ is the argument (angle between $r$ and $x$–axis) of the complex number $z = x + yi$. 

..............................
*Relationships among $x$, $y$, $r$ and $\emptyset$*

$x = r \cos \emptyset$; $y = r \sin \emptyset$; $r = \sqrt{x^2 + y^2}$; $\tan \emptyset = \frac{y}{x}$, if $x \neq 0$.

**Trigonometric function** The ratios of different pairs of sides of a right triangle are called *trigonometric functions* of its acute angle. There are six different ratios (or trigonometric functions) with following names: *sine*, *cosine*, *tangent*, *cotangent*, *secant* and *cosecant*.

**Trigonometric identities** Knowing one of the trigonometric functions of an acute angle, it is possible, by applying the relations (identities) given below, to determine the others. However, their main value lies in the possibility of substantially simplifying the aspect of many general formulas and thus reducing the computational process.

**Fundamental Identities**

1. $\sin^2 \alpha + \cos^2 \alpha = 1$;
2. $\tan \alpha \cdot \cot \alpha = 1$;
3. $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$;
4. $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$;
5. $\sin \alpha \cdot \csc \alpha = 1$;
6. $\cos \alpha \cdot \sec \alpha = 1$;
7. $\sec^2 \alpha = 1 + \tan^2 \alpha$;
8. $\csc^2 \alpha = 1 + \cot^2 \alpha$;
9. $\sin^2 \alpha = \frac{1}{1 + \cot^2 \alpha} = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1 - \cos 2\alpha}{2}$;
10. $\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{\cot^2 \alpha}{1 + \cot^2 \alpha} = \frac{1 + \cos 2\alpha}{2}$;
11. $\tan \alpha + \cot \alpha = \frac{1}{\sin \alpha \cos \alpha}$ \Rightarrow $\frac{\tan^2 \alpha + 1}{\tan \alpha} = \frac{1}{\sin \alpha \cos \alpha}$

**Expressing One Trigonometric Function in Terms of Another**

<table>
<thead>
<tr>
<th></th>
<th>sin</th>
<th>cos</th>
<th>tan</th>
<th>cot</th>
<th>csc</th>
<th>sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin</td>
<td>$\pm \sqrt{1 - \cos^2 x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\frac{1}{\tan x}$</td>
<td>$\frac{1}{\cot x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\pm \sqrt{\sec^2 x - 1}$</td>
</tr>
<tr>
<td>cos</td>
<td>$\pm \sqrt{1 - \sin^2 x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\frac{1}{\tan x}$</td>
<td>$\frac{1}{\cot x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\pm \sqrt{\csc^2 x - 1}$</td>
</tr>
<tr>
<td>tan</td>
<td>$\frac{\sin x}{\pm \sqrt{1 - \cos^2 x}}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\frac{1}{\cot x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\pm \sqrt{\sec^2 x - 1}$</td>
</tr>
<tr>
<td>cot</td>
<td>$\frac{\cos x}{\pm \sqrt{1 - \sin^2 x}}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\frac{1}{\tan x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\pm \sqrt{\csc^2 x - 1}$</td>
</tr>
<tr>
<td>csc</td>
<td>$\frac{1}{\sin x}$</td>
<td>$\frac{1}{\tan x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\pm \sqrt{\csc^2 x - 1}$</td>
</tr>
<tr>
<td>sec</td>
<td>$\frac{1}{\cos x}$</td>
<td>$\frac{1}{\tan x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\frac{1}{\sec x}$</td>
<td>$\frac{1}{\csc x}$</td>
<td>$\pm \sqrt{\csc^2 x - 1}$</td>
</tr>
</tbody>
</table>

**Trigonometric functions of sum/difference of two angles**
1. \( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \);
2. \( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \);
3. \( \tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \);
4. \( \cot(\alpha \pm \beta) = \frac{\cot \alpha \cot \beta \mp 1}{\cot \beta \pm \cot \alpha} \).

### Trigonometric functions of double-angle (double-angle identities)

1. \( \sin 2\alpha = 2 \sin \alpha \cos \alpha \);
2. \( \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \) or \( \cos 2\alpha = 2\cos^2 \alpha - 1 \) or \( \cos 2\alpha = 1 - 2\sin^2 \alpha \);
3. \( \tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha} \);
4. \( \cot 2\alpha = \frac{\cot^2 \alpha - 1}{2\cot \alpha} \).

### Trigonometric function of half-angle (half-angle identities)

1. \( \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \);
2. \( \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \);
3. \( \tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha} \);
4. \( \cot \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{\sin \alpha}{1 - \cos \alpha} = \frac{1 + \cos \alpha}{\sin \alpha} \).

These identities, which show the relationships between trigonometric functions, hold true for the trigonometric functions of any angle (or for all values of the angle \( \alpha \)).

### Trigonometric Functions of \( 0^\circ, 30^\circ, 45^\circ, 60^\circ \) and \( 90^\circ \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \sin \alpha )</th>
<th>( \cos \alpha )</th>
<th>( \tan \alpha )</th>
<th>( \cot \alpha )</th>
<th>( \sec \alpha )</th>
<th>( \csc \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( 30^\circ ) or ( \frac{\pi}{6} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>( \sqrt{3} )</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
<td>2</td>
</tr>
<tr>
<td>( 45^\circ ) or ( \frac{\pi}{4} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>1</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>( 60^\circ ) or ( \frac{\pi}{3} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>( \sqrt{3} )</td>
<td>2</td>
<td>( \frac{2\sqrt{3}}{3} )</td>
</tr>
<tr>
<td>( 90^\circ ) or ( \frac{\pi}{2} )</td>
<td>1</td>
<td>0</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>1</td>
</tr>
</tbody>
</table>

**Trigonometry** Trigonometry* is the study of how the sides and angles of a triangle are related to each other. The most important mathematical contributions of famous Persian polymath and prolific writer...
Nasir al-Din al-Tusi (1201 – 1274) was the creation of trigonometry as a mathematical discipline.

*The word “trigonometry” is derived from two Greek words, “trigonon”, triangle, and “metria”, measure. The basic task of trigonometry is the solution of triangles, finding unknown quantities of a triangle from given values of other of its quantities. To solve triangle means to know all three sides and all three angles. Such, for example, is the problem of computing the angles of a triangle from given sides, computing the sides and one angle of a triangle from the area and two angles, etc. Since any computational problem of geometry may be reduced to the solution of triangles, trigonometry finds applications in the entire field of plane and solid geometry, and is extensively used in many areas of natural science and engineering. The angles of an arbitrary triangle cannot be connected with its sides by means of algebraic relations. For this reason, trigonometry introduces new quantities in addition to the angles themselves: these are the so-called trigonometric functions or trigonometric ratios, which can be connected with the sides of the triangle by simple algebraic relations (ratios).

Trichotomy law If \(a\) and \(b\) are two real numbers, then one of the following possibilities can be true:
1) \(a = b\), 2) \(a > b\), 3) \(a < b\).

trinomial A polynomial containing exactly three terms.

triple integral See “Multiple integral”.

trisecting an angle It is impossible to make this construction with straightedge and compass alone. This is one of the oldest unsolved problems in geometry.

truth* The state of being in accord/harmony with reality. Truth has a variety of the logical, factual, ethical and philosophical meanings**.

*Thomas Aquinas (1225-1274): “Truth is equation (or adequation) of things and intellects.”
Jiddy Krishnamurti (1895-1986): “Truth is pathless land.”
Omar Khayyam (1048-1131): “A hair divides what is false and true.”
Chief Cochise (1805 – 1874): “It does not require many words to speak the truth.”
Chief Sitting Bull (1831 – 1890): “It does not take many words to tell the truth.”
**Gottfried Wilchelm Leibniz (1646 – 1716): “There are also two kinds of truths: truth of reasoning and truths of fact. Truths of reasoning are necessary and their opposite is impossible; those of fact are contingent and their opposite is possible.”
Georg W. F. Hegel (1770 – 1831): The true is the whole.”
Michael Idvorski Pupin (1858 – 1935): Truth is beautiful and divine, no matter how humble its origin; it is the same in the musty boiler-room as it is in the glorious stars of heaven.”
Ernest Hemingway (1899 – 1961): “There’s no one thing that is true. They’re all true.”
Chief Seattle (1780 – 1866): “This We Know. All Things Are Connected.”

truth table A table that lists all possible combinations of truth values for a given statement or combination or statements.
turning points  The points on the graph of a function where the function changes its behavior from increasing to decreasing or inversely.

U

undefined quotient  A quotient of any non-zero number divided by zero.

union of two sets A and B  The union of two sets A and B, written A ∪ B, is the set of all members that are in A or B, i.e. \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \).

unit circle  A circle with center at origin of the coordinate plane and radius 1.

unit vector  A vector that has magnitude/length 1.

unit tangent vector  Let C be a smooth curve with position vector \( r(t) \). The unit tangent vector, denoted \( T(t) \) is defined to be

\[
T(t) = \frac{r'(t)}{|r'|}
\]

unlike terms  The terms with different variables or different exponents on the variables. For example, \( x^2 \) and \( x^3 \) are unlike terms.

upper quartile  The median of the upper half of an ordered set of data.

UY Scuti  UY Scuti is a bright red supergiant/hypergiant and pulsating variable star in the constellation Scutum. It is a leading candidate for being the largest known star. It has an estimated average median radius of 1,708 solar radii, thus a volume 5 billion times that of the Sun. It is approximately 2,900 parsecs (9,500 light-years) from Earth.

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V

valid argument  An argument is valid if that fact all the premises are true forces the conclusion to be true. An argument that is not valid is invalid, or a fallacy.

value  The numerical result after a number has been substituted into an expression.
value of an expression  When you are asked to find the value of an expression, that means you are looking for the result that you get when you evaluate the expression.

variable  A quantity which, within the framework of a given problem, takes on various values. In other words, variable is a letter that represents an unknown number /quantity or part of the rule that varies. Variables are mostly denoted by last letters of the alphabet: x, y, z.

variable costs  In business, costs that vary according to the amount of products produced.

variance  An another measure of dispersion of data (set of n numbers: \( x_1, x_2, \ldots, x_n \), with mean \( \bar{x} \)). It is square of the standard deviation \( s \), symbolized \( s^2 \). Thus,

\[
    s^2 = \frac{\sum x^2 - n\bar{x}^2}{n-1}.
\]

variation  A variation is a relation between sets of values of one variable and a set of values of other variables.

variation constant  The constant, \( k \), in an equation of direct \( y = kx \) or inverse variation \( y = \frac{k}{x} \); also called constant of proportionality.

variation in sign  A change from positive to negative or from negative to positive in successive terms of the polynomial when they are written in order of descending powers of the variable.

vector  A vector (geometrically) is a directed line segment. The length of the line segment represents a vector quantity or magnitude (a numerical measure). In other words, a vector is a quantity which involves direction and magnitude, and must have both for a full description. Examples of vector quantities are velocity, acceleration, force, a directed line segment in space, etc. Two vectors are equal if they are of the same length, lie on parallel or coinciding lines, and are in the same direction.

Vector Fields  (See Scalar and Vector Fields)

Vector (Cross) Product of Two Vectors  The vector (cross) product of a vector \( \mathbf{a} \) by a non-collinear vector \( \mathbf{b} \) (denoted by \( \mathbf{a} \times \mathbf{b} \)) is a third vector \( \mathbf{c} \) which satisfies the following three conditions:

\( (a) \) its length \( |\mathbf{c}| = |\mathbf{a}| |\mathbf{b}| \sin \theta \), where \( \theta \) is the angle between the vectors \( \mathbf{a} \) and \( \mathbf{b} \);

\( (b) \) it is perpendicular to the vectors \( \mathbf{a} \) and \( \mathbf{b} \), that is, \( \mathbf{c} \perp \mathbf{a} \) and \( \mathbf{c} \perp \mathbf{b} \);

\( (c) \) the three vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) taken in the indicated order, form a right-handed coordinate system;

\( (d) \) Definition of Vector/Cross Product

Let \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \) and \( \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \) be vectors in space.

The vector/cross product of \( \mathbf{a} \) and \( \mathbf{b} \) is the vector
\[ c = a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k = (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k \]

It is important to note that this definition applies only to three–dimensional vectors.

**velocity** The rate of change of distance with respect to time.

**vertex** The common endpoint of two rays that form an angle, or point at which the graph of a parabola, an ellipse, or hyperbola crosses its axis of symmetry.

**vertical angles** Two opposite angles formed by two intersecting lines. Vertical angles have equal measures.

**vertical asymptote** A straight line \( x = c \) is called a vertical asymptote to the graph of \ a function \( f(x) \) if \[ \lim_{x \to c^-} f(x) = \infty \] or \[ \lim_{x \to c^+} f(x) = \infty. \]

**Example** Find the vertical asymptotes to the graph of the function \( f(x) = \frac{5}{x^2 - 4} \).

**Solution** The denominator vanishes at the points \( x_1 = 2 \) and \( x_2 = -2 \) (solutions of the equation \( x^2 - 4 = 0 \)).

We have \[ \lim_{x \to \pm 2} f(x) = \lim_{x \to \pm 2} \frac{5}{x^2 - 4} = \infty, \] hence the straight lines \( x = -2 \) and \( x = 2 \) are vertical asymptotes.

**vertical-line test** The statement that a graph represents a function if it is impossible to draw a vertical line that intersects the graph more than once. In other words, if vertical line intersects a graph in at most one point, then the graph is that of a function.

**vigesimal number system** This ancient system of numbers is based on 20 – the total number of fingers and toes. The Maya used vigesimal number system. The Mayan numerals was in use in Mesoamerica by 1000 BC. The earliest explicit use of zero occurred on Maya’s monuments dated to 357 AD.

**volume** The size of a solid expressed in cubic (three-dimensional) units.

**volume of a solid of revolution** If \( f(x) \) is nonnegative and \( R \) is the region between \( f(x) \) and the x-axis from \( x = a \) to \( x = b \), the volume of the solid formed by rotating \( R \) about the x-axis is given by \[ V = \int_a^b \pi [f(x)]^2 dx. \]
**Weierstrass** functions  Weierstrass functions are famous for being continuous everywhere, but differentiable “nowhere”.

For example, \[ w(x) = \sum_{n=0}^{\infty} \frac{\sin (2^n x)}{2^n} = \sin x + \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \ldots \] is a Weierstrass function.

*Karl Theodor Wilhelm Weierstrass* (1815 – 1897), German mathematician and “father of modern mathematical analysis/calculus”. In Weierstrass’ original paper, the function was defined by

\[ f(x) = \sum_{n=0}^{\infty} a^n \cos (b^n \pi x), \]

where \( 0 < a < 1 \), \( b \) is odd natural number, and \( ab > 1 + \frac{3}{2} \pi \).

**Weierstrass theorem** Any monotonic bounded sequence has a limit.

**well defined set** A set is well defined if its contents can be clearly determined.

**whole numbers** The set of numbers \{0, 1, 2, 3, \ldots\} or the set of natural/counting numbers and 0.

**wind chill** A temperature reading that takes into account the chilling effect of the wind. Wind chill temperature is always lower than thermometer temperature because when the wind blows, your body feels colder than actual temperature.

**word names for numbers** To write whole number in words, we will begin at the left with the largest period. N. B.: For large numbers, digits are separated by commas, into groups of three, called periods. Write (or say) the number in each group of three, followed by the name for that group. When you get the ones group, do not include the group name. Hyphens (dashes) are used whenever you write only a number from 21 to 99.

**Example** Write a word name for the number 7,056,124.

**Solution** Seven million, fifty-six thousand, one hundred twenty-four.

**work** The concept of work is used especially when a force moves an object from one point to another. If a constant force \( F \) is applied to an object, moving it a distance \( d \) in the direction of the force, then the work \( W \) done on the object is given by equation \( W = Fd \).

**work problems** (See “Modeling Work Problems” and “amount of work completed”) A problem that involves finding the rate (or time) at which two or more people working together can finish a job.

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**X**

**x-axis** A reference horizontal line in a system of coordinates. This horizontal number line represents the first coordinates/components (x-values) of the ordered pairs.
\textbf{x – intercept}  The point on the x-axis where the graph of an function/relation intersects the x-axis. In other words, the x-intercept is the x-coordinate of an ordered pair where \( y = 0 \).

\textbf{Y}

\textbf{y-axis}  A reference vertical number line in a system of coordinates which represents the second coordinates/components (y-values) of the ordered pairs.

\textbf{year}  The time taken by a planet to make the revolution around the sun. One year means 12 months, 365 days, 8760 hours, 525,600 minutes, or 31,536,000 seconds.

\textbf{y – intercept}  The point on the y-axis where the graph of an function/relation intersects the y-axis. The y-intercept is the y-coordinate of an ordered pair where \( x = 0 \).

\textbf{Z}

\textbf{z-axis}  A third coordinate axis in the rectangular coordinate system in three dimensions, perpendicular to both the x- and y-axes at the origin.

\textbf{zero*}  The number between the set of all negative real numbers and the set of all positive real numbers. This amazing cardinal number, like any boundary, is so reach source of paradoxes, indicating the absence of quantity. For a lot mathematical people zero is \textit{all or nothing**}, or, precisely, mathematical \textit{nothing} and an \textit{everything} simultaneously. \textit{“Used unwisely, zero has the power to destroy logic”} (Charles Seife: \textit{“ZERO – The Biography of a Dangerous Idea”}, Penguin Books, 2000, pg. 219). But, if we deal with “nothing” and “everything” we are in the middle of the metaphysics.

*The Indian mathematician \textbf{Aryabhata} (476 – 550 AD) invented zero. To honor him Indian government named the first Indian satellite by his name. He used word “kha” for a position and it would be used later as the name for zero. \textbf{Brahmagupta} (598 – 668) was the next in the line of the great mathematicians from the classical age of Indian mathematics. He gave us the first rules for dealing with a zero as a number and for dealing with positive (“fortune”) and negative (“debt”) numbers.

**Today we don’t say “nothing” when we mean “zero”. Zero is something – it is a number, digit and place holder. That means nothing can be something – an \textbf{ambiguity}. But we can use ambiguity, contradiction, and paradox to create mathematics.

\textbf{zero exponent}  For any nonzero real number \( a \), \( \quad a^0 = 1 \).
**z-intercept** The point on z-axis where the graph of an function/relation intersects/crosses the z-axis. In other words, z-intercept is the z-coordinate of an ordered triple where \( x = 0 \), and \( y = 0 \).

**zero matrix** A matrix containing only zero elements is called zero matrix.

**zero of multiplicity \( n \)** A polynomial function has a zero (or root) \( x_0 \) of multiplicity \( n \) if \( x_0 \) occurs \( n \) times, that is, polynomial has \( n \) factors of \( x - k \). For example, \( f(x) = (x - 4)^5 \). If \( f(x) = 0 \), then \( (x - 4)^5 = 0 \) \( \Rightarrow \) \( x - 4 = 0 \) \( \Rightarrow \) \( x = 4 \). Therefore, \( 4 \) is a zero of \( f(x) \) of multiplicity 5.

**zero polynomial** The polynomial function of the zero degree with only constant term and defined by \( f(x) = 0 \) (i.e., the constant term is 0).

**zero-product/factor property** If \( ab = 0 \), then \( a = 0 \) or \( b = 0 \). In other words, if the product of two (or more) numbers/factors is 0, then at least one of the numbers/factors must be 0.

**zero’s properties** 1. **Addition property of 0**: Adding 0 to any number \( a \) leaves the number \( a \) unchanged, i.e. \( a + 0 = a \). 2. **Division by 0** is undefined. When 0 is divided by any nonzero number, the quotient (result) is 0. 3. **Multiplication property of 0**: Multiplying any number by 0 gives a product (result) of 0.

**z score** See standard score.

**zero vector** The zero vector is the vector with magnitude 0.

**zeros of the function \( f(x) \)** The \( x \)-values for which a function \( f(x) = 0 \). Geometrically speaking, zeros are \( x \)-intercepts (or roots or solutions of equation \( f(x) = 0 \)).

---

*“The first duty of government is to protect the powerless from the powerful.”*

Hammurabi, 6th king of Babylon (1810 BC – 1750 BC)

*“Three things cannot be long hidden: the sun, the moon, and the truth.”*

Buddha

*“To know that you do not know is the best. To pretend to know when you do not know is a disease.”*

Lao – Tzu (604 BC – 531 BC)

*There is nothing permanent except change.”*

Heraclitus (544 BC – 483 BC)

*“The only true wisdom is in knowing you know nothing.”*

Socrates (469 BC – 399 BC)
“I am indebted to my father for living, but to my teacher for living well.”
Alexander the Great (356 BC – 323 BC)

“The ink of the scholar is more holy than blood of the martyr.”
Prophet Muhammad (570? – 632)

“Violence never settles anything.”
Genghis Khan (1162 – 1227)

“Love all, trust a few, do wrong to none.”
William Shakespeare (1564 – 1616)

“The more you know, the less sure you are.”
Voltaire (1694 – 1778)

“An investment in knowledge pays the best interest.”
Benjamin Franklin (1706 – 1790)

“There is nothing worse than aggressive stupidity.”
Johann Wolfgang von Goethe (1749 – 1832)

“Change alone is eternal, perpetual, immortal.”
Arthur Schopenhauer (1788 – 1860)

“Wisdom and peace come when you start living the life the creator intended for you.”
Geronimo (1829 – 1909)

“The only thing we have to fear is fear itself.”
Franklin Roosevelt (1882 – 1945)

“The power to question is the basis of all human progress.”
Indira Gandhi (1917 – 1984)

“Education is a system of imposed ignorance.”
Noam Chomsky

“Learning is a treasure that will follow its owner everywhere.”
Chinese Proverb

“The enemy of art is the absence of limitations.”
Orson Welles (1915 – 1985)

“Nothing in all the world is more dangerous than sincere ignorance and conscientious stupidity.”
Martin Luther King, Jr. (1929 – 1968)

“Knowledge speaks, but wisdom listens.”
Jimi Hendrix (1942 – 1970)

“If you don’t like something, change it. If you can’t change it, change your attitude.”
Maya Angelou (1928 – 2014)
Mathlympics challenge for high schoolers

Mohave Community College faculty members, from far left, Don Plantz, Marko Rucnov (Coordinator of 14th Mathlympics) and Matt Butcher along with Academic Chair Steve Sorden and the chair’s secretary, Kathy Cooper, break out their Mathlympics T-shirts in preparation for the 2011 competition, taking place April 29 2011 at the Bullhead City campus. Mathlympics allows students to demonstrate and receive recognition for their outstanding math abilities. Students from River Valley, Mohave, Lake Havasu, Kingman and Laughlin high schools as well as area homeschooled students will take part. Each school is sending three-student teams to compete at one of five levels: Algebra I, Geometry, Algebra II, Trigonometry, and Mathematical Analysis/Calculus. Individual medals and team trophies will be awarded, along with a plaque for the overall winner. (Mohave Valley Daily News, 04/28/2011)
The word of author at the end

Metaphysics/Metamathematics as the refuge
ETERNAL RETURN

“A man is taken for his word and an ox for its horns.”

Serbian Proverb

After work in my College Algebra, Trigonometry or Calculus class I am always in good mood for a walk and contemplative pondering and speculation. In my everyday walks through the windy and sandy Mohave Desert I feel, as a rule, special mental and metaphysical pleasure and calm, so to say - nirvana. The eternal game between wind and sand over there is so metaphoric and poly-semantic. Wind was always wind and sand was always sand. And, without rest, the wind turned and plowed through the sand, but neither did the wind turn the sand over, nor did the sand tire the wind. Just like in eternal game between life and death, peace and war, or love and hate.

All these pessimistic stories, linked one after the other into the endless chain of human fate, result neither in despair, nor hopelessness. Hovering above the river of man’s existence, by adding up the zeros we can reach the infinity. Ivo Andric, the only Serb Nobel Prize winner was so right. For me, dividing by zero is the fastest way to reach that. The closest neighborhood with this question is the question of the all questions – How to reach and how to feel the eternal beauty of man’s endurance in the perpetual repetition of the same, and yet always slightly different? Again Andric and his “Ex Ponto” and “The Travnik Chronicles”. Everything is changing, I know. All things are in flux – “Panta rhei”, old Heraclitus had known that very well 2,500 years ago. Everything flows, meaning that everything is constantly changing, from the smallest grain of sand to the stars in the sky. Nobody can step in the same Colorado River two times. Its canyon shows us that so powerful in last few million years. I am smitten by that, but, as a mathematics instructor and mathematician, I can say of myself as one seeking original truth, the main root of all prosperities and human light. At the same time, as a writer, I can say of myself as one seeking original sin, the main root (together with aggressive ignorance) of all evils and human darkness, but in my books nothing begins with the first, or ends with the last page.
On the other hand, the central unknown in the equation of our lives is a fear. I have seen that, in these Wall Street’ times, the main and frequently the sole initiator of human action is fear – panicking, senseless, often groundless, but true and profound fear. Perhaps there were other motives in the beginning, but nowadays fear is the main one. The fear of failure is a special problem and major cause for our procrastinations. Fear makes people evil, and mean, it makes them generous, even good. The daring, wonderful soul of man lies dead like stone at the bottom of the sea, while his body is governed by the bestial fear and incomprehensible panic of nerves. The old story! Free thinker is always a newcomer everywhere: from one to another never land. He who is courageous and proud (I saw that many times and little bit more than that) loses his bread and freedom, his property and life, quickly and easily, but he who bends his head and gives in to fear, he loses so much of himself, he is so consumed by fear that his life becomes worthless, said wise Andric. Those who choose fear over love and, little by little, day after day, death over life. The way to conquer fear is to face it. Unfortunately, human actions influence always later events but not earlier events. Just like effects and causes. Namely, effects occur after but never (or almost never) before their causes. The universal law of cause and effects is visible everywhere. The effect or evidence is mostly what we can see and what experience, while the cause is often more far reaching and indirect. However, we can always trace the effect back to its cause, but who can follow unreadable tracks in the snow of our and global passing and fate. And, how in that we can reach the core of the coldness that warms one? Solution for this eternal problem may lie in old task since golden Pythagorean times - Be strong in mind and know thy-self, but we have to keep in mind also, at the same time, the old dilemma of my Bishop Rade – ’Who is a man, but has to be a man’, and an older dilemma of William Shakespeare – “We know what we are, but know not what we may be.” Actually that is our life crucified between hope and fear.

*George Polya, famous Hungarian American mathematician and father of mathematical problem solving (1887 – 1985): “To teach effectively a teacher must develop a feeling for his subject, he cannot make his students sense its vitality if he does not sense it himself. He cannot share his enthusiasm when he has no enthusiasm to share. How he make his point may be as important as the point he makes, he must personally feel it to be important.”
Marko Rucnov, Grand Canyon, October 2013
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Always and everywhere

Use

T A T A T A

think ask* tell and think again

Rule

(Therefore, it is your most important and best interest to be educated, informed and capable of critical thinking.)

*”Ask, and it will be given to you, seek, and you will find, knock, and it will be opened to you.”
Matthew 7:7

“Thoughts are like arrows: once released, they strike their mark. Guard them well or one day you may be your own victim.”
Navajo Proverb

“Teaching should come from within instead of without.”
Hopi Proverb

“Our first teacher is our own heart.”
Cheyenne Proverb

“After dark all cats are leopards.”

“Teaching is an art, delivered from the heart.”
Clark County School District, Nevada

Zuni Proverb

“Reserve your right to think, for even to think wrongly is better than not to think at all.”
Hypatia of Alexandria (370 AD – 415 AD)